SSE counter-example

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1 Introduction

In this section we shall define the assumptions, that we shall use for convenience. First of all, we shall assume that the Dirichlet series for Riemann zeta-function of the kind

$$\sum_{n=1}^{\infty} \tfrac{1}{n^s}, \Re(s) \in (0,1) \tag{1}$$

is summable by the means of analytic continuation. By this representation we mean the analytic continuation of the provided series, as given by the provided link (`https://www.0bq.com/se`). We shall follow the provided methods of proofs and support each of them with the link. Indeed, there are some inconveniences with the usage of this method, where we substitute the summation rules. It is quite fascinating to think that way, but we obtain the flawed results with dropping off some of the conventions. For being rigorous we shall cite each step with the corresponding link and by the way of contradiction we shall accept each of them to be true.

2 Auxilliary statements

First of all we shall use the representation of zeta-function, given as follows (ABC zeta-function).

Lemma 1. *This result is given here:* ***https://www.0bq.com/azf***

$$\zeta(s) = \sum_{n=1}^{N} \tfrac{1}{n^s} - \tfrac{N^{1-s}}{1-s} - s\int_{N}^{\infty} \tfrac{x-\lfloor x \rfloor}{x^{s+1}}.$$

As the integral vanishes after taking the limit $N \to \infty$, we shall further use the notation $o(1)$ for this integral and obtain the following representation, which is the notation of the term, which is convergent to zero.

$$\zeta(s) = \sum_{n=1}^{N} \tfrac{1}{n^s} - \tfrac{N^{1-s}}{1-s} + o(1).$$

Now we shall make an observation, which justifies our counter-example.

Lemma 2.

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{n^s} = \lim_{k\to\infty} \sum_{n=1}^{m(k)} \frac{1}{n^s}, \Re(s) \in (0,1).$$

where $\{m(k)|k \geq 1\}$ *is any other subsequence of the sequence of the positive integers.*

Proof. We know that it holds for the series (1) in the domain $Re(s) > 1$ as this series is absolutely convergent on the corresponding domain. Therefore, by the uniqueness theorem, as the domain $Re(s) > 1$ consists of the accumulation points, if the series (1) can be defined on the domain $\Re(s) \in (0,1)$, those limits should be equal on this domain to the same sum, like it is done before in the corresponding article `https://www.0bq.com/arslt` while performing the step from (8) to (9). □

3 Counter-example

The idea of the counter-example is to imply from SSE to results, which contradict each other. The proof of one of those is given by Aric Behzad Cannanie, so we shall not provide the proof of it in the current paper.

Lemma 3. *Let* s *be a non-trivial zero of Riemann zeta-function. Then SSE* $\Longrightarrow \Re(s) = \frac{1}{2}$.

Proof. The proof may be found by the link `https://www.0bq.com/arslt`, the result corresponds to the equation (15). □

Lemma 4. *Let* s *be a non-trivial zero of Riemann zeta-function. Then SSE* $\Longrightarrow \Re(s) = \frac{1}{3}$.

By the Lemma 1 we obtain the following.

$$\zeta(s) = \sum_{n=1}^{8k^3} \frac{1}{n^s} - \frac{(8k^3)^{1-s}}{1-s} + o(1), \tag{2}$$

$$\zeta(1-\bar{s}) = \sum_{n=1}^{8k^6} \frac{1}{n^{(1-\bar{s})}} - \frac{(8k^6)^{\bar{s}}}{\bar{s}} + o(1). \tag{3}$$

We note that $\{8k^3|k \geq 1\}$ and $\{8k^6|k \geq 1\}$ are the subsequences of the sequence of positive integers such that $\lim_{k\to\infty} 8k^3 = \lim_{k\to\infty} 8k^6 = \infty$. Therefore, we are able to use Lemma 2. Subtract (3) from (2) and obtain the following with respect to Lemma 2 after taking the limit as $k \to \infty$.

$$\zeta(s) - \zeta(1-\bar{s}) = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^{(1-\bar{s})}} + \lim_{k\to\infty} (\frac{(8k^6)^{\bar{s}}}{\bar{s}} - \frac{(8k^3)^{1-s}}{1-s}). \tag{4}$$

If s is a non-trivial zero of Riemann zeta-function, we obtain the following with respect to SSE from (4).

$$\lim_{k\to\infty}\left(\frac{8^{\bar{s}}k^{6\bar{s}}}{\bar{s}} - \frac{8^{1-s}k^{3(1-s)}}{1-s}\right) = 0. \tag{5}$$

Now we use the trick, which is given while deriving (15) in `https://www.0bq.com/arslt`. For the limit of this difference to be equal to zero one should have the same order of those terms, i.e., the real part of the degrees of k should be the same as $\forall z \in \mathbb{C}|e^z| = e^{\Re(z)}$ as otherwise one of the terms blows up. Therefore, we obtain

$$6\Re(s) = 3(1-\Re(s)) \iff 2\Re(s) = 1-\Re(s) \iff \Re(s) = \frac{1}{3}.$$

Thus, the proof is complete. Now for convenience we check the identity (5) for $s = \frac{1}{3}$. We obtain

$$\lim_{k\to\infty}\left(\frac{8^{\frac{1}{3}}k^{6\frac{1}{3}}}{\frac{1}{3}} - \frac{8^{1-\frac{1}{3}}k^{3(1-\frac{1}{3})}}{1-\frac{1}{3}}\right) = \lim_{k\to\infty}(6k^2 - 6k^2) = 0.$$

Now we shall show that the only possibility for SSE to be correct is the complete absence of non-trivial zeroes of Riemann zeta-function.

Theorem 1. *If SSE is correct, then Riemann zeta-function has got no non-trivial zeroes.*

Proof. Assume that SSE holds. By the way of contradiction, let us assume that there exists a non-trivial zero for the Riemann zeta-function and denote it as s. Then by the Lemma 3 we obtain $\Re(s) = \frac{1}{2}$. But at the same time, by the Lemma 4 we obtain $\Re(s) = \frac{1}{3}$. Thus, $\frac{1}{2} = \Re(s) = \frac{1}{3} \implies \frac{1}{2} = \frac{1}{3}$, which is a contradiction. Thus, the theorem is proven. □

Thus, from the Theorem 1 we obtain that any non-trivial zero of Riemann zeta-function is actually a numerical counter-example to SSE. Those zeroes may be found in the corresponding section of the following article. `https://en.wikipedia.org/w/index.php?title=Riemann_zeta_function&section=4&oldid=1194635919`