THE COUNTER-EXAMPLE TO SUPER SYMMETRY EQUATION

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Abstract

Dear Aric, after a poorly slept night I have come up with idea of constructing counter-example to your statement. Moreover, I shall avoid the moments of which we argued since we have not come to understanding. Hope this will satisfy you. I decided not to use the fancy geometry, but the basic knowledge of complex analysis, which you've got.

1 Introduction

For this argument I shall use the Riemann Functional Equation, which you could simply find in Titchmarch's book or in the artice in Wikipedia, which refers to the book anyway. The Riemann Functional Equation is the following:

$$\zeta(s) = 2^s \pi^{s-1} \sin \tfrac{\pi s}{2} \Gamma(1-s)\zeta(1-s).$$

Also I shall use the Schwarz Reflection principle, which states the following:

$$\forall f \in A(\Omega) d \in \mathbb{R} : d \in \Omega, f(d) \in \mathbb{R} \implies \forall z \in \Omega f(\bar{z}) = f(\bar{z}).$$

Moreover I would use ABC zeta-function and for the simplicity of notation I would sign the integral part as $o(1)$.

2 Studying the limit part of equation

In this section we would state that $\lim_{k\to\infty} \sum_{n=1}^{k} (\frac{1}{n^s} - \frac{1}{n^{1-\bar{s}}}) = 0 \implies \Re(s) = \frac{1}{2}$.

Lemma 1. *The Super Symmetric equation holds only for* $\Re(s) = \frac{1}{2}$.

Proof. Without loss of generality let us assume that $1 - \Re s > \Re s$ since with the substitution $s' = 1 - s$ we would obtain the similar contradiction construction for the opposite inequality. Let us write ABC zeta-function for s and $1 - \bar{s}$:

$$\zeta(s) = \sum_{n=1}^{k} \tfrac{1}{n^s} - \tfrac{k^{1-s}}{1-s} + o(1),$$

$$\zeta(1-\bar{s}) = \sum_{n=1}^{k} \tfrac{1}{n^{1-\bar{s}}} - \tfrac{k^{\bar{s}}}{\bar{s}}.$$

Now let us subtract both equations with respect to the condition $\zeta(s)-\zeta(1-\bar{s}) = 0$ and put the sums to the Left Hand Side:

$$\sum_{n=1}^{k} \tfrac{1}{n^s} - \sum_{n=1}^{k} \tfrac{1}{n^{1-\bar{s}}} = \tfrac{k^{1-s}}{1-s} - \tfrac{k^{\bar{s}}}{\bar{s}} + o(1).$$

Since $1-\Re(s) > 0$ we may deduce $\lim_{k\to\infty} |\frac{k^{1-s}}{1-s}| = +\infty$. Then as $\lim_{k\to\infty} \sum_{n=1}^{k}(\frac{1}{n^s} - \frac{1}{n^{1-\bar{s}}}) = 0$ we conclude that $\sum_{n=1}^{k}(\frac{1}{n^s} - \frac{1}{n^{1-\bar{s}}}) = o(\frac{k^{1-s}}{1-s})$. From the assumption $1-\Re(s) > \Re(s)$ we obtain $\frac{k^{\bar{s}}}{\bar{s}} = o(\frac{k^{1-s}}{1-s})$. Now let us divide both parts of this equality by $\frac{k^{1-s}}{1-s}$ and take the limit of our equality. With respect to our previous observations we obtain:

$$0 = 1,$$

which is a contradiction. Hence $\Re(s) = \frac{1}{2}$. □

Notice that Lemma 1 was also proven in your video on The Proof Of Riemann Hypothesis, so you can't argue with that. Conversely, let us prove that $\Re(s) = \frac{1}{2} \implies \lim_{k\to\infty} \sum_{n=1}^{k}(\frac{1}{n^s} - \frac{1}{n^{1-\bar{s}}}) = 0$, which is understood by you as $\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^{1-\bar{s}}} = 0$ according to your videos and the reasoning on your website.

Lemma 2. $\Re(s) = \frac{1}{2} \implies \lim_{k\to\infty} \sum_{n=1}^{k}(\frac{1}{n^s} - \frac{1}{n^{1-\bar{s}}}) = 0.$

Proof. Let us notice that for $\Re(s) = \frac{1}{2}$ if we set $s = d+it$ we would conclude the following: $s = \frac{1}{2} + it$ and $1-\bar{s} = 1-(\bar{s}) = 1-(\overline{\frac{1}{2}+it}) = 1-\frac{1}{2}+it = s$. Hence the wanted sequence becomes the constant zero sequence, which is convergent to zero as the constant zero sequence. □

3 Reformulating the problem

In the previous section we have shown the equivalence $\Re(s) = \frac{1}{2} \iff \lim_{k\to\infty}(\sum_{n=1}^{k} \frac{1}{n^s} - \sum_{n=1}^{k} \frac{1}{n^{1-\bar{s}}}) = 0$, since we have proven the two-sided implication by Lemma 1 and Lemma 2. By the transitivity of equivalence it is enough to show that the equivalence $\zeta(s) - \zeta(1-\bar{s}) = 0 \iff \Re(s) = \frac{1}{2}$ is false. For the needs of this section we shall introduce the following function:

$$\Phi(s) := 2^s \pi^{s-1} \sin \tfrac{\pi s}{2} \Gamma(1-s),$$

which we shall refer to. Notice that Riemann zeta-function has got no real zeroes in the critical strip, which makes the following argument possible.

Lemma 3. *$\Phi(s)$ and $\Phi(1-s)$ are multiplicative inverses.*

Proof. Let us express $\Phi(s)$ with respect to Riemann functional equation and use it again for $\zeta(1-s)$:

$$\zeta(s) = \Phi(s)\zeta(1-s) = \Phi(s)\Phi(1-s)\zeta(s) \iff \Phi(s)\Phi(1-s) = 1$$

□

Now we would look for our counter-example as a curve, which does not lie on the critical line, but also satisfies the wanted identity. For this purpose we would need one lemma, which justifies the counter-example we would find further.

Lemma 4. *Suppose that the function $f : (-L, L) \to \mathbb{C}, 0 < L \leq \pi$ is analytical in some neighbourhood of the real interval and satisfies the following functional equation:*

$$f(t) = -e^{it} f(-t).$$

Then f is identical zero function.

Proof. Consider function f as an element of Hilbert space $L_2((-\pi, \pi))$ by continuing it outside of $(-L, L)$ by zero, if it is needed. Then because it is analytical, its Fourier series would be convergent at the domain $(-L, L)$ to the function f homogeneously and its Fourier series admits the termwise differentiation and the termwise derivative of the Fourier series of F is convergent to the derivative of f. Now let us represent the function f as follows:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}.$$

By substituting this into the original equation we observe:

$$\sum_{n=-\infty}^{\infty} c_n e^{int} = -\sum_{n=-\infty}^{\infty} c_n e^{i(1-n)t},$$

$$\sum_{n=-\infty}^{\infty} c_n (e^{int} + e^{i(1-n)t}) = 0.$$

Divide this equation by $e^{\frac{it}{2}}$ and apply the Euler's formula for representation of cosine function:

$$\sum_{n=-\infty}^{\infty} c_n (e^{i(n-\frac{1}{2})t} + e^{i(\frac{1}{2}-n)t}) = 0,$$

$$2 \sum_{n=-\infty}^{\infty} c_n \cos{(n-\tfrac{1}{2})t} = 0,$$

$$\sum_{n=-\infty}^{\infty} c_n \cos{(n-\tfrac{1}{2})t} = 0.$$

Since the system $\cos{(n-\frac{1}{2})t}|_{n\in\mathbb{Z}}$ is linear independent we deduce that $\forall n \in \mathbb{Z} c_n = 0$, which proves the lemma statement. □

4 Building the counter-example

Lemma 4 introduced us the sufficient condition of our curve to be the curve of counter-examples. However, it is not necessary, which I point to avoid the discussion of "counter-examples" to my statement. For the simplicity we shall study the equation $\Phi(s) = e^{i\alpha}$ for some $\alpha \in \mathbb{R}$. Let us study this equation from the geometric perspective.

Lemma 5. *The equation $\Phi(s) = e^{i\alpha}, \alpha \in U \subset \mathbb{R}$ defines at least one analytical curve on the complex plain $s(\alpha)$, where U is some real symmetric interval, containing* 0*.*

Proof. Let us rewrite this equation in the following form:

$$F(\alpha, s) := e^{-i\alpha}\Phi(s) - 1 = 0.$$

Take the derivative of F with respect to s:

$$\tfrac{d}{ds}F(\alpha, s) = e^{-i\alpha}\Phi^{'}(s).$$

Since $\Phi(s)$ is a non-constant analytical function, the zeroes of $\Phi^{'}(s)$ would be a set of izolated points. Therefore in the neighbourhood of any point on the complex plain we can find a point, where $\Phi^{'}(s) \neq 0$. Hence, by the Implicit Function Theorem we obtain the statement of the lemma.

Notice that this Lemma could be proven in another way, but with the slight usage of mathematical engines. It is easy to check that $|\pi\Phi^{'}(\frac{1}{2})| > 15$, which means that $\Phi^{'}$ is non-zero in some neighbourhood of $\frac{1}{2}$. Therefore, due to the Lagrange Inversion Theorem, there exists a local inverse $G : \Phi(V) \to V$, where V is a neighbourhood of $\frac{1}{2}$, where $\Phi^{'}(s) \neq 0$. Since $e^{i0} = 1 = \Phi(\frac{1}{2})$, by the continuity of $e^{i\alpha}$ there would exist the containing zero real interval $L \subset \mathbb{R}$: $e^{iL} \subset \Phi(V)$ and hence the curve $s(\alpha) = G(e^{i\alpha}), \alpha \in L$ is well defined. □

The next thing we would like to show is that the real component of this curve is non-constant.

Lemma 6. *Let $s(\alpha) = l(\alpha) + it(\alpha)$ be a curve, defined by the equation $\Phi(s) = e^{i\alpha}, \alpha \in L$ such that $s(0) = \frac{1}{2}$. Then $l(\alpha) \neq const$.*

Proof. By the way of contradiction let us assume that $l(\alpha) = \frac{1}{2} = const$. Let us take the derivative of this equation with respect to α:

$$it^{'}(\alpha)\Phi^{'}(\tfrac{1}{2} + it(\alpha)) = ie^{i\alpha},$$

$$it^{'}(\alpha)\Phi^{'}(\tfrac{1}{2} + it(\alpha)) = i\Phi(\tfrac{1}{2} + it(\alpha)),$$

$$t^{'}(\alpha)\Phi^{'}(\tfrac{1}{2} + it(\alpha)) = \Phi(\tfrac{1}{2} + it(\alpha)).$$

Take $t(\alpha)$ to be an odd parameterization of the imaginary part of the curve, since $e^{-i\alpha}$ is conjugated to $e^{i\alpha}$ and it preserves conjugation by the Schwarz

Reflection Principle. Then by the Schwarz reflection principle $\Im[\Phi(\frac{1}{2}+it(\alpha))]$ is an odd function. But the derivative of this function with respect to $it(\alpha)$, i.e. $\Im[\Phi^{'}(\frac{1}{2}+it(\alpha))]$ should be even as the derivative of an odd function. Therefore $t^{'}(\alpha)\Phi^{'}(\frac{1}{2}+it(\alpha))$ is an even function as the product of two even functions since $t^{'}(\alpha)$ is even as the derivative of an odd function. This means that $\Im[\Phi(\frac{1}{2}+it(\alpha))]$ is even and odd at the same time, which is only possible for the constant zero function. But $\Im[\Phi(\frac{1}{2}+it(\alpha)))] = \sin\alpha \neq 0$ constantly by the construction, hence we obtain a contradiction. This means that $l(\alpha) \neq const$.

In addition let us prove separately two facts that we have used: that the zero is the only function, which is odd and even at the same time and that the derrivative of an odd function is even.

Let us prove that zero is the only function, which is odd and even at the same time. Suppose that some non-zero function f is odd and even at the same time. From evenness we obtain the following representation:

$$f(z) = \frac{f(z)+f(-z)}{2}.$$

On the other hand we can obtain from the oddness:

$$f(z) = \frac{f(z)-f(-z)}{2}.$$

Now let us subtract the second equation from the first and obtain:

$$0 = f(-z).$$

Since it is true for any z we would obtain that f is a constant zero function.

Now let us suppose that f is an odd function. Then by the definition we obtain:

$$f(z) = \frac{f(z)-f(-z)}{2}.$$

After differentiating we obtain by the chain rule:

$$f^{'}(z) = \frac{f^{'}(z)+f^{'}(-z)}{2},$$

which means that $f^{'}$ is even by the definition. □

Therefore we have obtained that there exists a curve of counter-examples along the curve from *Lemma*6.

Lemma 7. *Let $s(\alpha)$ be a curve from the Lemma 6. Then $s^{'}(\alpha) \neq 0$ and $\Phi^{'}(s(\alpha)) \neq 0$ for all $\alpha \in L$, where L is the same as in the proof of Lemma 5.*

Proof. Take the derivative of both parts with respect to α of both parts of the equation $\Phi(s(\alpha)) = e^{i\alpha}$ and obtain:

$$s^{'}(\alpha)\Phi^{'}(s(\alpha)) = ie^{i\alpha}.$$

Since the exponential function is never zero and the product is zero if one of the multiples is zero, we obtain the statement of the lemma. □

The Lemma 7 guarantees the existence of at least one analytical local inverse function to $\Phi(s)$ by the Lagrange Inversion Theorem. Then it would preserve our curve and we could find the closed formula for the parametrization of the curve from Lemma 6 as follows:

$$s(\alpha) = G(e^{i\alpha}), \alpha \in L. \tag{1}$$

Now it is left to check that this curve satisfies the condition $\zeta(s(\alpha)) - \zeta(1 - \bar{s}(\alpha)) = 0$, which can be rewritten as follows with respect to our parametrization as follows.

$$\zeta(s(\alpha)) - \zeta(1 - s(-\alpha)) = 0.$$

Denote $f(\alpha) = \zeta(s(\alpha)) - \zeta(1-s(-\alpha))$ and prove that it should be a constant zero function. Note that f is an analytical function as a composition of analytical functions. From Lemma 3 and the condition $|\Phi(s(\alpha))| = 1$ we deduce that $\Phi(1 - s(\alpha)) = \Phi(s(-\alpha))$ by the uniqueness of multiplicative inverse. Now let us write the Riemann functional equation for $\zeta(s(\alpha))$ and $\zeta(1 - s(-\alpha))$ with respect to our parametrization:

$$\zeta(s(\alpha)) = e^{i\alpha}\zeta(1 - s(\alpha)),$$

$$\zeta(1 - s(-\alpha)) = e^{i\alpha}\zeta(s(-\alpha)).$$

Subtract the second equation from the first and obtain:

$$f(\alpha) = -e^{i\alpha}f(-\alpha).$$

Since f satisfies the equation from Lemma 4, it is a constant zero function, which guarantees that along this curve $\zeta(s) - \zeta(1 - \bar{s}) = 0$ is satisfied. The numerical counter-example then could be represented as follows as this curve does not lie on the critical line by Lemma 6.

$$s(\arg_{(-\pi,\pi)} \max |\Re s(\alpha) - \tfrac{1}{2}|)$$

5 The numeical counter-example

We are able to implement this approach with the following Python code.

```
import numpy as np
from mpmath import mp, findroot, zeta
from scipy import pi

mp.dps = 25

def Phi(s): #computing Phi
    return 2**s * mp.pi**(s-1) * mp.sin(pi*s/2) * mp.gamma(1-s)

def Phi_inv(x): #finding the inverse to Phi
```

```
    def equation(s):
        return Phi(s) - x
    sol = findroot(equation, 0.5-1j, solver='muller')
    return sol


x = np.linspace(-pi, pi, 12345) #Take 12345 points on the interval (0, pi)
Phi_inv_values = [Phi_inv(np.exp(1j * xi)) for xi in x] #compute the values of P
max_distance_index = np.argmax(np.abs(np.real(Phi_inv_values) - 1 / 2))#look for
max_distance_s = Phi_inv_values[max_distance_index]#computing s

result = zeta(max_distance_s) - zeta(1 - max_distance_s.conjugate())#check the v

print("Zeta diff=", result)
print("s=", max_distance_s)
```

As the outcome we obtain the following.

```
Zeta diff= (1.5478849147379941699420740e-16 + 8.2159416375863195720664910e-17j)
s= (0.500000000000000005076299623 - 4.01958209432810972571006 8j)
```

As you see, the difference of zeta-functions is extremely low. We use the Muller's algorithm to construct the inverse function. You can see that the deviation from the critical line is extremely high. If you were correct in your conclusions, then the Muller's algorithm would leave the real part of the variable untouched, since by the iteration form as we have supposed in our initial approximation that the real part of the variable is 0.5. See the quadratic equation that runs the Muller's recurrence formula. Therefore, if you were correct, we would not have obtained even the smallest stray. And just for controlling we have computed the value of your difference in that point and saw that it is extremely close to zero. But of course, we could not obtain the precise value, since we have computed the approximation of zeta-function in the approximation of zero of the desired function.

6 What went wrong

The problem is that you do not understand the concept of analytic continuation. The analytic continuation does not imply that you are able to study the divergent representation of some analytic function. To be the most accurate, let me show one counter-example, where such claim ruins all of the mathematics, if it is correct. In my most recent preprint on the Riemann Hypothesis I have shown that the series

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{s+\frac{3}{4}}}l_k(\bar{\theta}_0),$$

where $l_n(\bar{\theta}) = \prod_{p\in P} e^{-i2\pi m_p(n)\theta^{(q(p))}}$, P is the set of all prime numbers, $q(p)$ is the number of the prime number p in the ordered set of prime numbers and $m_p(n)$ is the multiplicity of the prime number p in the factorization of n, $\bar{\theta}_0 = (\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, ...)$ can be rearranged to be convergent to any analytic function in the space $A(|z| < r)$ in this domain. We also know that for $\Re(s) > \frac{1}{4}$ this series is absolutely convergent. As you know, the rearrangement o the absolutely convergent series does not influence this sum. Therefore, by your logic, using the Identity theorem (of course, not the correct one, but your understanding of it), we deduce that any analytic function is equal to any other analytic function, which is nonsense. Stop mixing up the notions of the function and the representation of the function!