Lemma 1 and 2 appear to be correct in their statements, but the proofs may have some flaws. There is no need for an additional proof, and you can state the result as a direct consequence of RSLT (not SSE).

Lemma 3. *$\Phi(s)$ and $\Phi(1-s)$ are multiplicative inverses.*

This Lemma 3 appears to be correct.

ζ(s)/ ζ(1-s)* ζ(1-s)/ ζ(s)

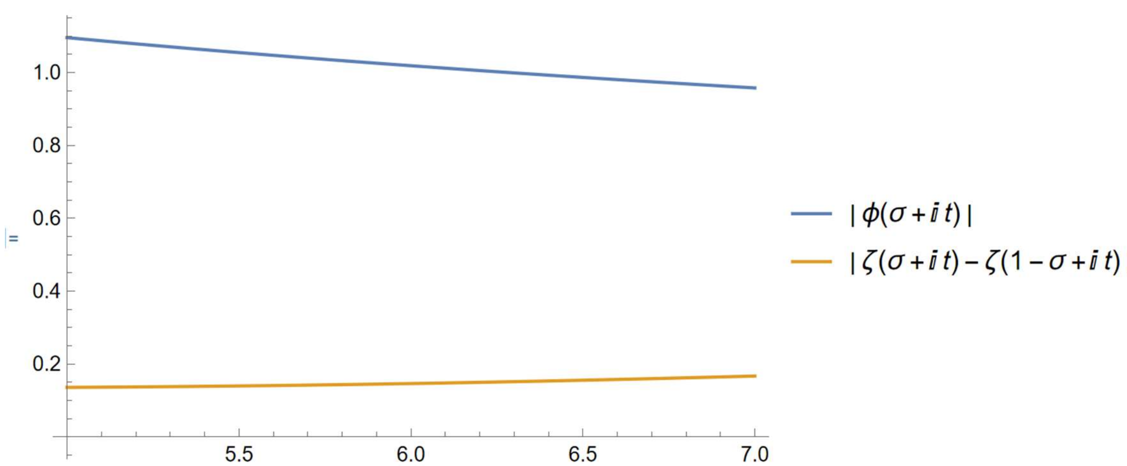
$\sqcup$

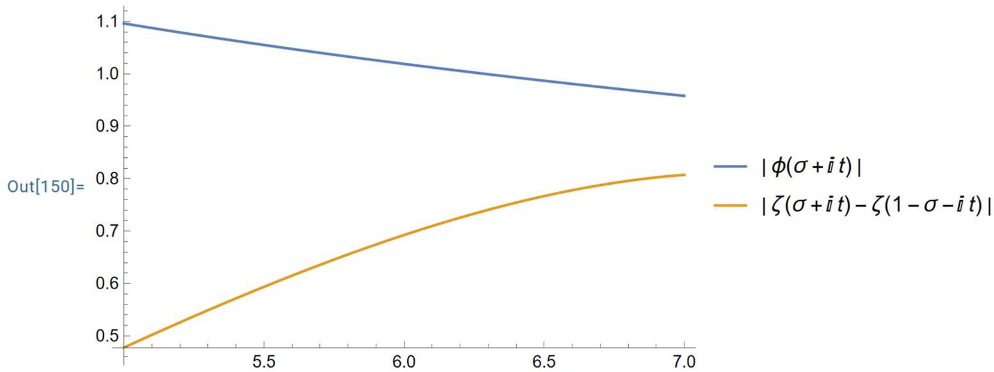
Lemma 4. *Let $\epsilon > 0$ be the imaginary part of the closest to the real line point, where $\zeta(s) = 0$ in the critical strip. Then $\zeta(s) - \zeta(1-\bar{s}) = 0 \iff |\Phi(s)| = 1$ in the domain $\{s|\Re(s) \in (0,1), \Im(s) \in (-\epsilon, \epsilon)$.*

$t \in (5, 7)$

$\sigma = .9$

Numeric Counterexample





Let me explain what the numerical counterexample for Lemma 4 means. You can see a descending blue line that starts above 1 and ends below it. At the same time, you can see an orange line above zero, which proves that at some point ϕ(s) equals one while ζ(s) - ζ(1-s*) is not zero. Therefore, Lemma 4 is false, and all the subsequent lemmas that are based on it are also false.

We cannot assume that ε is zero simply because it is very close to zero. As you stated, ε > 0 meaning that ε ≠ 0, so we need to be consistent and acknowledge that ε can be zero or nonzero.

According to your definition for the zeros of zeta functions, either |ζ(s+ε)-ζ(1-s+ε)| > 0 or |ζ(s+ε)-ζ(1-s-ε)|≠0 and |Φ(s+ε)| ≠1. Additionally, Φ'(s) is not equal to Φ'(s+ε). For example, consider the derivative of |x| at zero, which is undefined (according to mathematicians). At -ε, it is -1, and at ε, it is 1.

Lemma 4. *Let $\epsilon > 0$ be the imaginary part of the closest to the real line point, where $\zeta(s) = 0$ in the critical strip. Then $\zeta(s) - \zeta(1-\bar{s}) = 0 \iff |\Phi(s)| = 1$ in the domain $\{s|\Re(s) \in (0,1), \Im(s) \in (-\epsilon, \epsilon), \Phi^4(s) - 1 \neq 0\}$.*

New version of lemma 4 is False:

$$\Phi(s)^4 - 1 \neq 0 \Leftrightarrow \Phi(s)^4 \neq 1 \Leftarrow \Phi(s) \neq \pm 1 \Leftarrow |\Phi(s)| \neq 1.$$

$$\Phi(s)^4 - 1 = 0 \Leftrightarrow \Phi(s)^4 = 1 \Rightarrow \Phi(s) = \pm 1 \Rightarrow |\Phi(s)| = 1.$$

$\Phi(s) \in \mathbb{C}$. |Φ(s)| cannot be and not equal to 1 simultaneously.

The response below is not acceptable, and no further communication on this matter is recommended.
"|Φ(s)| = 1 does not imply Φ^4(s) = 1, so this is correctly defined. If you remember some basics of geometry from the middle school, you know that cos^2(x) + sin^2(x) = 1 and hence Φ(s) = e^{i α} is well-defined and it does not necessarily satisfy this polynomial equation."

The absolute value of 1 does not mean that the 4-th power would be one. Please, try cos(\pi/16) + i sin(\pi /16). The square of the absolute value here is cos^2(\pi/16) + sin^2(\pi/16) = 1, but due to the deMoivre's formula the fourth power is the following:

cos(π/4) + i sin(π/4) = \sqrt(2)/2 + i \sqrt(2)/2 \neq 1. You may find all of the needed information here:

https://en.m.wikipedia.org/wiki/De_Moivre%27s_formula

Above shows that proves $1/2 + 1/2 = 1$. In other words, it shows that $|\Phi(s)|$= 1 does not imply $\Phi(s) = 1$ and has no relevance to this topic. Lemma 4 is false.

Note that $|\Phi(s)^4| = |\Phi(s)| = 1$

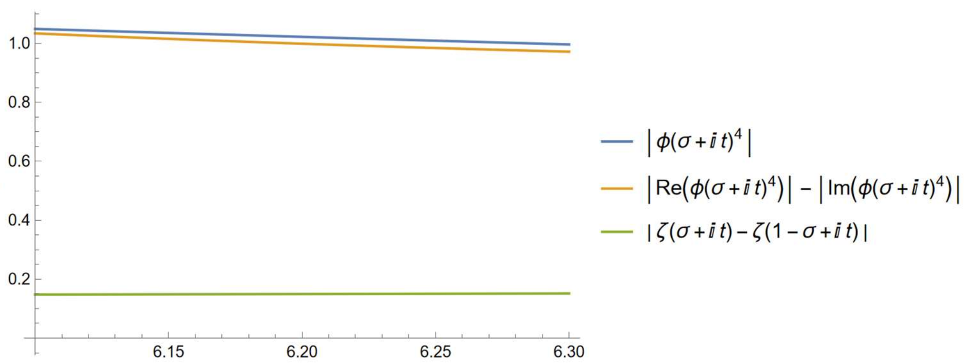
Below is numerical counterexample that lemma 4 doesn't hold.

t ∈ (6.1, 6.3)

$\sigma = .9$

$\Phi(s)^4 - 1 \neq 0$ and $|\Phi(s)^4| = 1$

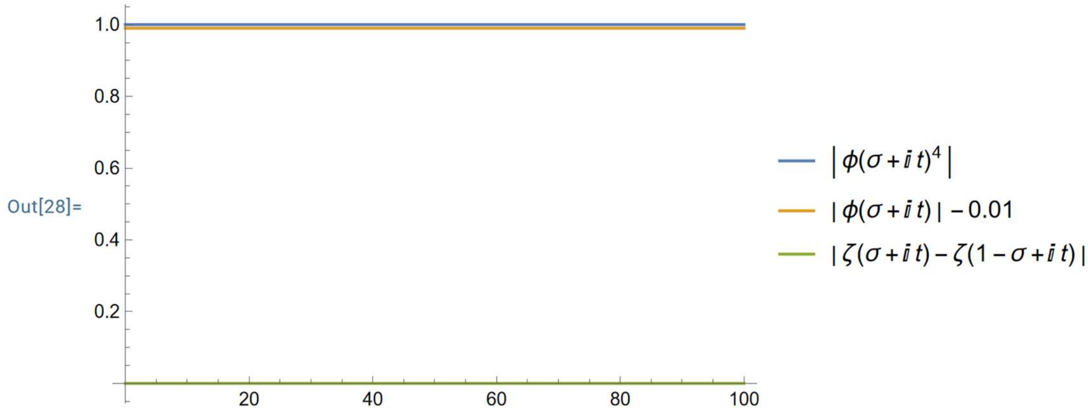
Out[48]=



Here is the computational proof I have. I intentionally subtracted 0.01, and I will not respond to any variation of the proof. It will not lead us anywhere, and it is not an efficient use of our time. However, please keep in mind that we have no obligation to read or review your paper. At this point, I am only looking for a numerical counterexample.

t ∈ (0, 100)

$\sigma = .5$



Lemma 5. *The equation $\Phi(s) = e^{i\alpha}, \alpha \in \mathbb{R}$ defines at least one analytical curve on the complex plain $s(\alpha)$.*

Proof. Let us rewrite this equation in the following form:

$$F(\alpha, s) := e^{-i\alpha}\Phi(s) - 1 = 0.$$

Take the derivative of F with respect to s:

$$\tfrac{d}{ds}F(\alpha, s) = e^{-i\alpha}\Phi'(s).$$

Since $\Phi(s)$ is a non-constant analytical function, the zeroes of $\Phi'(s)$ would be a set of izolated points. Therefore in the neighbourhood of any point on the complex plain we can find a point, where $\Phi'(s) \neq 0$. Hence by the Implicit Function Theorem we obtain the statement of the lemma. □

The next thing we would like to show is that the real component of this curve is non-constant.

$$\frac{d}{ds}e^{-i\alpha}\Phi(s) - 1 = 0 \quad \Rightarrow \quad e^{-i\alpha}\Phi'(s) = 0$$

Assmuing Φ' (s)≠0 that means $e^{-i\alpha}$=0.

You cannot use any s you want; you must use the condition |Φ(s)|=1. If you use a different s because Φ'(s)=0, let's say s1, there is no reason to assume that |Φ(s1) |=1.

If you want to say that we are studying the critical strip regardless of $|\Phi(s)|=1$, then you have no reason to say in Lemma 6 that the curve $s(\alpha)$ must be on the critical line.

You as you are not using it correctly The Implicit Function Theorem and neighborhoods. For example, please consider the function $F(x,y) = x^2 + y^2 - 1$. And let me know why you think that it's satisfied at the points $(0, \pm 1)$ because it's satisfied in the neighborhood?

Let $F(x,y)=x^2+y^2-1$ and the implicit function theorem is not satisfied at the points $(0,\pm 1)$

Lemma 6. *Let $s(\alpha) = l(\alpha) + it(\alpha)$ be a curve, defined by the equation $\Phi(s) = e^{i\alpha}, \alpha \in (-\pi, \pi)$ such that $s(0) = \frac{1}{2}$. Then $l(\alpha) \neq const.$*

Lemma 6 states that l(α) cannot be constant, including the value of 1/2. This leaves us with two possibilities:

1. The lemma is referring to a path that has no direct connection to ζ(s) = ζ(1-s). If this is the case, then the relevance of the lemma is unclear.
2. The lemma is implying that ζ(s) & ζ(1-s) cannot be equal on any straight line including the critical line. As far we know all non-trivial zeros of the zeta function lie on the critical line and ζ(s) = ζ(1-s) in that line.

Therefore, we can conclude that Lemma 6 is either irrelevant or incorrect in SSE context.

There is no requirement for the path or analytical curve of α to be constant unless you show in lemma 5.

According to Lemma 5 $e^{i\alpha}\Phi(\mathrm{s}) = 1$ and Lemma 3 $\Phi(\mathrm{s})\,\Phi(1-\mathrm{s}) = 1$ thus $e^{i\alpha} = \Phi(1-\mathrm{s})$

Also α=$-i\ ln\ (\Phi(\mathrm{s}))$

Because Φ(s) is none constant analytical function therefore Φ(1-s) is non-constant function.

$$it^{'}(\alpha)\Phi^{'}(\tfrac{1}{2} + it(\alpha)) = ie^{i\alpha},$$

$$it^{'}(\alpha)\Phi^{'}(\tfrac{1}{2} + it(\alpha)) = i\Phi(\tfrac{1}{2} + it(\alpha)),$$

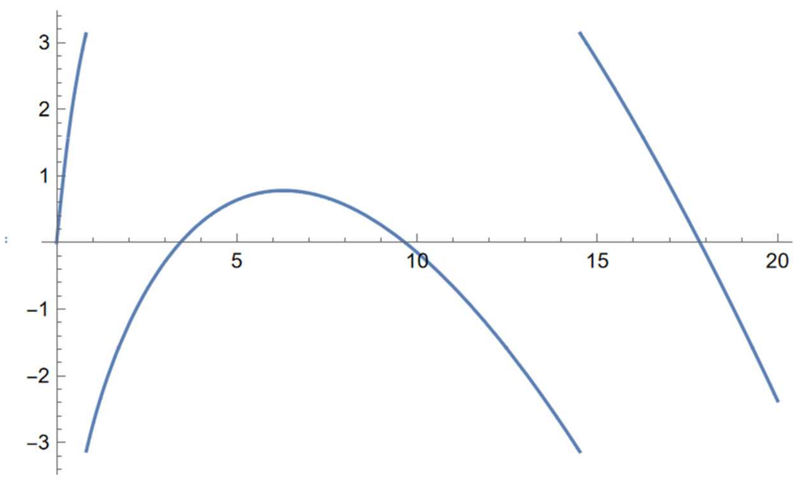
$$t^{'}(\alpha)\Phi^{'}(\tfrac{1}{2} + it(\alpha)) = \Phi(\tfrac{1}{2} + it(\alpha)).$$

Let's define g(s)=f(u) then we have g(s)-f(u) =0 take divertive d/du give us s'g'(s)-f'(u)=0 because RHS is zero that means s'g'(s)-f'(u) is even and odd and there are no contractions.

Take $t(\alpha)$ to be an odd parameterization of the imaginary part of the curve, since $e^{-i\alpha}$ is conjugated to $e^{i\alpha}$ and it preserves conjugation by the Schwarz Reflection Principle. Then by the Schwarz reflection principle $\Im[\Phi(\frac{1}{2}+it(\alpha))]$ is an odd function. But the derivative of this function with respect to $it(\alpha)$, i.e. $\Im[\Phi^{'}(\frac{1}{2}+it(\alpha))]$ should be even as the derivative of an odd function. Therefore $t^{'}(\alpha)\Phi^{'}(\frac{1}{2}+it(\alpha))$ is an even function as the product of two even functions since $t^{'}(\alpha)$ is even as the derivative of an odd function. This means that $\Im[\Phi(\frac{1}{2}+it(\alpha))]$ is even and odd at the same time, which is only possible for the constant zero function. But $\Im[\Phi(\frac{1}{2}+it(\alpha)))] = \sin\alpha \neq 0$ constantly by the construction, hence we obtain a contradiction. This means that $l(\alpha) \neq const.$ □

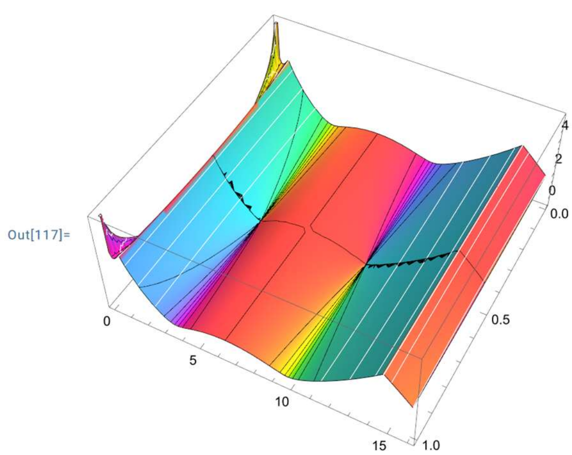
Lemma 7. *Let $s(\alpha)$ be a curve from the Lemma 6. Then $s^{'}(\alpha) \neq 0$ and $\Phi^{'}(s(\alpha)) \neq 0$ for all $\alpha \in (-\pi, \pi)$.*

α is not a continuous function, and you cannot take derivative of s (α) with respect to α. Also, you cannot use any chain rule to differentiate s (α) implicitly. I plotted α(s) =-I log(Φ(s)) on critical line. As stated in Lemma 7 α ∈ (−π, π) which that means every time you reach α=π it must jump to – π and vice versa.



Furthermore, you have specified that α belongs to an open interval, excluding the points π and -π. This implies that α is not continuous at those points, which means that α'(s) or s'(α) does not exist at π and -π.

Contour shows that there is no Analytical path for α




```
In[123]:= ep = .01;
          Z = N[ZetaZero[1]];

In[125]:= Alfa[s_] := -I Log[si[s]];
          AbsArg[Alfa[Z + .1 + I ep]]
          AbsArg[Alfa[Z - .1 - I ep]]

Out[126]= {2.83476, 3.11297}

Out[127]= {2.81855, -3.11286}
```