THE COUNTER-EXAMPLE TO SUPER SYMMETRY EQUATION

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Abstract

Dear Aric, after a poorly slept night I have come up with idea of constructing counter-example to your statement. Moreover, I shall avoid the moments of which we argued since we have not come to understanding. Hope this will satisfy you. I decided not to use the fancy geometry, but the basic knowledge of complex analysis, which you've got.

1 Introduction

For this argument I shall use the Riemann Functional Equation, which you could simply find in Titchmarch's book or in the artice in Wikipedia, which refers to the book anyway. The Riemann Functional Equation is the following:

$$\zeta(s) = 2^s \pi^{s-1} \sin \tfrac{\pi s}{2} \Gamma(1-s)\zeta(1-s).$$

Also I shall use the Schwarz Reflection principle, which states the following:

$$\forall f \in A(\Omega) d \in \mathbb{R} : d \in \Omega, f(d) \in \mathbb{R} \implies \forall z \in \Omega f(\bar{z}) = f(\bar{z}).$$

Moreover I would use ABC zeta-function and for the simplicity of notation I would sign the integral part as $o(1)$.

2 Studying the limit part of equation

In this section we would state that $\lim_{k\to\infty} \sum_{n=1}^{k} (\frac{1}{n^s} - \frac{1}{n^{1-\bar{s}}}) = 0 \implies \Re(s) = \frac{1}{2}$.

Lemma 1. *The Super Symmetric equation holds only for* $\Re(s) = \frac{1}{2}$.

Proof. Without loss of generality let us assume that $1 - \Re s > \Re s$ since with the substitution $s' = 1 - s$ we would obtain the similar contradiction construction for the opposite inequality. Let us write ABC zeta-function for s and $1 - \bar{s}$:

$$\zeta(s) = \sum_{n=1}^{k} \tfrac{1}{n^s} - \tfrac{k^{1-s}}{1-s} + o(1),$$

$$\zeta(1-\bar{s}) = \sum_{n=1}^{k} \frac{1}{n^{1-\bar{s}}} - \frac{k^{\bar{s}}}{\bar{s}}.$$

Now let us subtract both equations with respect to the condition $\zeta(s)-\zeta(1-\bar{s}) = 0$ and put the sums to the Left Hand Side:

$$\sum_{n=1}^{k} \frac{1}{n^s} - \sum_{n=1}^{k} \frac{1}{n^{1-\bar{s}}} = \frac{k^{1-s}}{1-s} - \frac{k^{\bar{s}}}{\bar{s}} + o(1).$$

Since $1-\Re(s) > 0$ we may deduce $\lim_{k\to\infty} |\frac{k^{1-s}}{1-s}| = +\infty$. Then as $\lim_{k\to\infty} \sum_{n=1}^{k}(\frac{1}{n^s} - \frac{1}{n^{1-\bar{s}}}) = 0$ we conclude that $\sum_{n=1}^{k}(\frac{1}{n^s} - \frac{1}{n^{1-\bar{s}}}) = o(\frac{k^{1-s}}{1-s})$. From the assumption $1-\Re(s) > \Re(s)$ we obtain $\frac{k^{\bar{s}}}{\bar{s}} = o(\frac{k^{1-s}}{1-s})$. Now let us divide both parts of this equality by $\frac{k^{1-s}}{1-s}$ and take the limit of our equality. With respect to our previous observations we obtain:

$$0 = 1,$$

which is a contradiction. Hence $\Re(s) = \frac{1}{2}$. □

Notice that Lemma 1 was also proven in your video on The Proof Of Riemann Hypothesis, so you can't argue with that. Conversely, let us prove that $\Re(s) = \frac{1}{2} \implies \lim_{k\to\infty} \sum_{n=1}^{k}(\frac{1}{n^s} - \frac{1}{n^{1-\bar{s}}}) = 0$, which is understood by you as $\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^{1-\bar{s}}} = 0$ according to your videos and the reasoning on your website.

Lemma 2. $\Re(s) = \frac{1}{2} \implies \lim_{k\to\infty} \sum_{n=1}^{k}(\frac{1}{n^s} - \frac{1}{n^{1-\bar{s}}}) = 0$.

Proof. Let us notice that for $\Re(s) = \frac{1}{2}$ if we set $s = d+it$ we would conclude the following: $s = \frac{1}{2} + it$ and $1-\bar{s} = 1-(\bar{s}) = 1-(\overline{\frac{1}{2}+it}) = 1-\frac{1}{2}+it = s$. Hence the wanted sequence becomes the constant zero sequence, which is convergent to zero as the constant zero sequence. □

3 Reformulating the problem

In the previous section we have shown the equivalence $\Re(s) = \frac{1}{2} \iff \lim_{k\to\infty}(\sum_{n=1}^{k} \frac{1}{n^s} - \sum_{n=1}^{k} \frac{1}{n^{1-\bar{s}}}) = 0$, since we have proven the two-sided implication by Lemma 1 and Lemma 2. By the transitivity of equivalence it is enough to show that the equivalence $\zeta(s) - \zeta(1-\bar{s}) = 0 \iff \Re(s) = \frac{1}{2}$ is false. For the needs of this section we shall introduce the following function:

$$\Phi(s) := 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s),$$

which we shall refer to. Notice that Riemann zeta-function has got no real zeroes in the critical strip, which makes the following argument possible.

Lemma 3. *$\Phi(s)$ and $\Phi(1-s)$ are multiplicative inverses.*

Proof. Let us express $\Phi(s)$ with respect to Riemann functional equation and use it again for $\zeta(1-s)$:

$$\zeta(s) = \Phi(s)\zeta(1-s) = \Phi(s)\Phi(1-s)\zeta(s) \iff \Phi(s)\Phi(1-s) = 1$$

□

Now we would consider the zero-free domain of the zeta-function in the neighbourhood of the point $\frac{1}{2}$. Notice that since Riemann zeta-function has got no real zeroes in the critical strip, we are able to consider such a small neighbourhood, since the zeroes of non-constant analytical functions are isolated points due to the Theorem of Uniqueness.

Lemma 4. *Let $\epsilon > 0$ be the imaginary part of the closest to the real line point, where $\zeta(s) = 0$ in the critical strip. Then $\zeta(s) - \zeta(1-\bar{s}) = 0 \iff |\Phi(s)| = 1$ in the domain $\{s|\Re(s) \in (0,1), \Im(s) \in (-\epsilon, \epsilon), \Phi^4(s) - 1 \neq 0\}$.*

Proof. Firstly, let us prove the implication $\zeta(s) - \zeta(1-\bar{s}) = 0 \implies |\Phi(s)| = 1$. Since $\zeta(s) = \zeta(1-\bar{s})$ we may deduce that $|\zeta(s)| = |\zeta(1-\bar{s})| = |\zeta(1-s)|$ according to the Schwarz reflection principle. Now let us write the Riemann functional equation and take the absolute value of the both parts:

$$|\zeta(s)| = |\Phi(s)||\zeta(1-s)| = |\Phi(s)||\zeta(s)|.$$

Since $|\zeta(s)| \neq 0$ in the considered domain, divide both parts by $|\zeta(s)|$ and obtain $|\Phi(s)| = 1$.

Let us now prove the implication $|\Phi(s)| = 1 \implies \zeta(s) - \zeta(1-\bar{s}) = 0$. Let us write the Riemann functional equation for $\zeta(\bar{s})$ considering $\Phi(1-\bar{s}) = \Phi(s)$ and $\Phi(1-s) = \Phi(\bar{s})$ by the uniqueness of multiplicative inverses:

$$\zeta(s) = \Phi(s)\zeta(1-s), \tag{1}$$

$$\zeta(1-\bar{s}) = \Phi(s)\zeta(\bar{s}). \tag{2}$$

Subtract (2) from (1) and obtain and notice that $\Phi(s) = \frac{\zeta(s)}{\zeta(1-s)} = \frac{\zeta(1-\bar{s})}{\zeta(\bar{s})}$:

$$\zeta(s) - \zeta(1-\bar{s}) = \Phi(s)[\zeta(1-s) - \zeta(\bar{s})].$$

Notice that $|\Phi(s)| = 1 \implies \Phi(\bar{s}) = \frac{1}{\Phi(s)}$. Multiply both parts of this equation by $\sqrt{\Phi(\bar{s})}$ and denote $\Psi(s) := \sqrt{\Phi(\bar{s})}[\zeta(s) - \zeta(1-\bar{s})]$. Then this equality becomes the following:

$$\Psi(s) = -\Psi(\bar{s}). \tag{3}$$

From this we conclude that $\Psi(s)$ is purely imaginary with respect to the Schwarz reflection principle and the fact that complex conjugation preserves sums and products. Let us square both sides of this equality and obtain:

$$\Phi(\bar{s})[\zeta(s) - \zeta(1-\bar{s})]^2 = \Phi(s)[\zeta(1-s) - \zeta(\bar{s})]^2.$$

Multiply and divide the Right Hand Side by $\Phi^2(\bar{s})$

$$\Phi(\bar{s})[\zeta(s) - \zeta(1-\bar{s})]^2 = \Phi^3(s)[\zeta(s) - \zeta(1-\bar{s})]^2,$$

$$(\Phi^4(s) - 1)[\zeta(s) - \zeta(1-\bar{s})]^2 = 0.$$

Notice that the case when $\Phi^4(s) = 1$ could be excluded by the assumption of the lemma. Therefore the equivalence is proven.

□

4 Building the counter-example

Lemma 4 introduced us the equivalence $\zeta(s) - \zeta(1-\bar{s}) = 0 \iff |\Phi(s)| = 1$ for some neighbourhood of the real part of the critical strip, which does not contain zeroes of Riemann zeta-function. By the polar representation of the complex numbers we need to study the values of the variable, which can satisfy the equation $\Phi(s) = e^{i\alpha}$ for some $\alpha \in \mathbb{R}$. Let us study this equation from the geometric perspective.

Lemma 5. *The equation $\Phi(s) = e^{i\alpha}, \alpha \in U \subset \mathbb{R}$ defines at least one analytical curve on the complex plain $s(\alpha)$, where U is some real interval, containing 0.*

Proof. Let us rewrite this equation in the following form:

$$F(\alpha, s) := e^{-i\alpha}\Phi(s) - 1 = 0.$$

Take the derivative of F with respect to s:

$$\tfrac{d}{ds}F(\alpha, s) = e^{-i\alpha}\Phi^{'}(s).$$

Since $\Phi(s)$ is a non-constant analytical function, the zeroes of $\Phi^{'}(s)$ would be a set of izolated points. Therefore in the neighbourhood of any point on the complex plain we can find a point, where $\Phi^{'}(s) \neq 0$. Hence by the Implicit Function Theorem we obtain the statement of the lemma.

Notice that this Lemma could be proven in another way, but with the slight usage of mathematical engines. It is easy to check that $|\pi\Phi^{'}(\frac{1}{2})| > 15$, which means that $\Phi^{'}$ is non-zero in some neighbourhood of $\frac{1}{2}$. Therefore due to the Lagrange Inversion Theorem, there exists a local inverse $G : \Phi(V) \to V$, where V is a neighbourhood of $\frac{1}{2}$, where $\Phi^{'}(s) \neq 0$. Since $e^{i0} = 1 = \Phi(\frac{1}{2})$, by the continuity of $e^{i\alpha}$ there would exist the containing zero real interval $L \subset \mathbb{R}$: $e^{iL} \subset \Phi(V)$ and hence the curve $s(\alpha) = G(e^{i\alpha}), \alpha \in L$ is well defined. □

The next thing we would like to show is that the real component of this curve is non-constant.

Lemma 6. *Let $s(\alpha) = l(\alpha) + it(\alpha)$ be a curve, defined by the equation $\Phi(s) = e^{i\alpha}, \alpha \in L$ such that $s(0) = \frac{1}{2}$. Then $l(\alpha) \neq const$.*

Proof. By the way of contradiction let us assume that $l(\alpha) = \frac{1}{2} = const$. Let us take the derivative of this equation with respect to α:

$$it^{'}(\alpha)\Phi^{'}(\tfrac{1}{2} + it(\alpha)) = ie^{i\alpha},$$

$$it^{'}(\alpha)\Phi^{'}(\tfrac{1}{2} + it(\alpha)) = i\Phi(\tfrac{1}{2} + it(\alpha)),$$

$$t^{'}(\alpha)\Phi^{'}(\tfrac{1}{2} + it(\alpha)) = \Phi(\tfrac{1}{2} + it(\alpha)).$$

Take $t(\alpha)$ to be an odd parameterization of the imaginary part of the curve, since $e^{-i\alpha}$ is conjugated to $e^{i\alpha}$ and it preserves conjugation by the Schwarz Reflection Principle. Then by the Schwarz reflection principle $\Im[\Phi(\frac{1}{2} + it(\alpha))]$ is an odd function. But the derivative of this function with respect to $it(\alpha)$, i.e. $\Im[\Phi^{'}(\frac{1}{2} + it(\alpha))]$ should be even as the derivative of an odd function. Therefore $t^{'}(\alpha)\Phi^{'}(\frac{1}{2} + it(\alpha))$ is an even function as the product of two even functions since $t^{'}(\alpha)$ is even as the derivative of an odd function. This means that $\Im[\Phi(\frac{1}{2} + it(\alpha))]$ is even and odd at the same time, which is only possible for the constant zero function. But $\Im[\Phi(\frac{1}{2} + it(\alpha)))] = \sin\alpha \neq 0$ constantly by the construction, hence we obtain a contradiction. This means that $l(\alpha) \neq const$.

In addition let us prove separately two facts that we have used: that the zero is the only function, which is odd and even at the same time and that the derrivative of an odd function is even.

Let us prove that zero is the only function, which is odd and even at the same time. Suppose that some non-zero function f is odd and even at the same time. From evenness we obtain the following representation:

$$f(z) = \tfrac{f(z)+f(-z)}{2}.$$

On the other hand we can obtain from the oddness:

$$f(z) = \tfrac{f(z)-f(-z)}{2}.$$

Now let us subtract the second equation from the first and obtain:

$$0 = f(-z).$$

Since it is true for any z we would obtain that f is a constant zero function.

Now let us suppose that f is an odd function. Then by the definition we obtain:

$$f(z) = \tfrac{f(z)-f(-z)}{2}.$$

After differentiating we obtain by the chain rule:

$$f^{'}(z) = \tfrac{f^{'}(z)+f^{'}(-z)}{2},$$

which means that $f^{'}$ is odd by the definition. □

Therefore we have obtained that either there exists a curve of counter-examples or $\Phi^4(s) = 1$ along the curve from *Lemma*6. But $\Phi^4(s) = 1$ along the continuous smooth curve is a contradiction, since that would mean that $\Phi^4(s) = 1$ in the whole complex plane according to the Theorem of Uniqueness.

Lemma 7. *Let $s(\alpha)$ be a curve from the Lemma 6. Then $s^{'}(\alpha) \neq 0$ and $\Phi^{'}(s(\alpha)) \neq 0$ for all $\alpha \in L$, where L is the same as in the proof of Lemma 5.*

Proof. Take the derivative of both parts with respect to α of both parts of the equation $\Phi(s(\alpha)) = e^{i\alpha}$ and obtain:

$$s^{'}(\alpha)\Phi^{'}(s(\alpha)) = ie^{i\alpha}.$$

Since the exponential function is never zero and the product is zero if one of the multiples is zero, we obtain the statement of the lemma. □

The Lemma 7 guarantees the existence of at least one analytical inverse function to $\Phi(s)$ by the Lagrange Inversion Theorem. Then it would preserve our curve and we could find the closed formula for the parameterization of the curve from Lemma 6 as follows:

$$s(\alpha) = G(e^{i\alpha}), \alpha \in L. \tag{4}$$

Notice that (4) defines an analytical curve without intersection with itself as a composition of injective functions on the wanted interval. Hence we are able to construct our counter-example as follows, which can be approximated with big enough computation powers:

$$s(\arg_{\alpha \in \Pi_s} \max |\Re s(\alpha) - \tfrac{1}{2}|),$$

where $\Pi_s := \{\alpha | s(\alpha) \in \{s | \Re(s) \in (0,1), \bar{\Im}(s) \in (-\epsilon, \epsilon)\}, \alpha \in (-\pi, \pi)\}$ and ϵ is the smallest positive imaginary part of the non-trivial zero of Riemann zeta-function.

5 The numeical counter-example

Dear Aric, it took me a few days to experiment with Wolfram Alpha to find a counter-example just by trying and studying the small neighbourhood of the point $\frac{1}{2}$. And I believe I have found it to be approximately the following:

$$\tfrac{1}{2} + 10^{-16.25561977} + \tfrac{i}{11^{16}}.$$

It gives a small decimal non-zero approximation for this number, but it gives exactly the same result for the other different numbers, which are small enough. But this number is what I found and it is the only one, which has got 0 in the column "alternative representation" after substituting in the wanted difference. Therefore I believe it is the best approximation of the wanted counter-example. I have done tones of plotting to understand the behaviour of zeta-function in that point and I am sure that it is fit to be the numerical counter-example. I shall send you the graphs I obtained and show you the asymptotic behaviour of zeta-function there. Please, let me know if you find some problems in the proof and shall we stay glued to SSE or try the ratio approach I suggested. You see, there are too many evidences for SSE being incorrect considering what I see, but I would be happy if it turns out that I have overlooked something. Otherwise I would be happy to complete your contest, since it allows me not to starve to death and buy a ring for my girlfriend and merry this amazing girl...)