THE COUNTER-EXAMPLE TO SUPER SYMMETRY EQUATION

Artem Afanasiev

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Abstract

Dear Aric, after a poorly slept night I have come up with idea of constructing counter-example to your statement. Moreover, I shall avoid the moments of which we argued since we have not come to understanding. Hope this will satisfy you. I decided not to use the fancy geometry, but the basic knowledge of complex analysis, which you've got.

1 Introduction

For this argument I shall use the Riemann Functional Equation, which you could simply find in Titchmarch's book or in the artice in Wikipedia, which refers to the book anyway. The Riemann Functional Equation is the following:

$$\zeta(s) = 2^s \pi^{s-1} \sin \tfrac{\pi s}{2} \Gamma(1-s)\zeta(1-s).$$

Also I shall use the Schwarz Reflection principle, which states the following:

$$\forall f \in A(\Omega) d \in \mathbb{R} : d \in \Omega, f(d) \in \mathbb{R} \implies \forall z \in \Omega f(\bar{z}) = \bar{f(z)}.$$

Moreover I would use ABC zeta-function and for the simplicity of notation I would sign the integral part as $o(1)$.

2 Studying the limit part of equation

In this section we would state that $\lim_{k\to\infty} \sum_{n=1}^{k} (\frac{1}{n^s} - \frac{1}{n^{1-\bar{s}}}) = 0 \implies \Re(s) = \frac{1}{2}$.

Lemma 1. *The Super Symmetric equation holds only for* $\Re(s) = \frac{1}{2}$.

Proof. Without loss of generality let us assume that $1 - \Re s > \Re s$ since with the substitution $s' = 1 - s$ we would obtain the similar contradiction construction for the opposite inequality. Let us write ABC zeta-function for s and $1 - \bar{s}$:

$$\zeta(s) = \sum_{n=1}^{k} \tfrac{1}{n^s} - \tfrac{k^{1-s}}{1-s} + o(1),$$

$$\zeta(1-\bar{s}) = \sum_{n=1}^{k} \frac{1}{n^{1-\bar{s}}} - \frac{k^{\bar{s}}}{\bar{s}}.$$

Now let us subtract both equations with respect to the condition $\zeta(s) - \zeta(1-\bar{s}) = 0$ and put the sums to the Left Hand Side:

$$\sum_{n=1}^{k} \frac{1}{n^s} - \sum_{n=1}^{k} \frac{1}{n^{1-\bar{s}}} = \frac{k^{1-s}}{1-s} - \frac{k^{\bar{s}}}{\bar{s}} + o(1).$$

Since $1 - \Re(s) > 0$ we may deduce $\lim_{k\to\infty} |\frac{k^{1-s}}{1-s}| = +\infty$. Then as $\lim_{k\to\infty} \sum_{n=1}^{k} (\frac{1}{n^s} - \frac{1}{n^{1-\bar{s}}}) = 0$ we conclude that $\sum_{n=1}^{k} (\frac{1}{n^s} - \frac{1}{n^{1-\bar{s}}}) = o(\frac{k^{1-s}}{1-s})$. From the assumption $1 - \Re(s) > \Re(s)$ we obtain $\frac{k^{\bar{s}}}{\bar{s}} = o(\frac{k^{1-s}}{1-s})$. Now let us divide both parts of this equality by $\frac{k^{1-s}}{1-s}$ and take the limit of our equality. With respect to our previous observations we obtain:

$$0 = 1,$$

which is a contradiction. Hence $\Re(s) = \frac{1}{2}$. □

Notice that Lemma 1 was also proven in your video on The Proof Of Riemann Hypothesis, so you can't argue with that. Conversely, let us prove that $\Re(s) = \frac{1}{2} \implies \lim_{k\to\infty} \sum_{n=1}^{k} (\frac{1}{n^s} - \frac{1}{n^{1-\bar{s}}}) = 0$, which is understood by you as $\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^{1-\bar{s}}} = 0$ according to your videos and the reasoning on your website.

Lemma 2. $\Re(s) = \frac{1}{2} \implies \lim_{k\to\infty} \sum_{n=1}^{k} (\frac{1}{n^s} - \frac{1}{n^{1-\bar{s}}}) = 0.$

Proof. Let us notice that for $\Re(s) = \frac{1}{2}$ if we set $s = d + it$ we would conclude the following: $s = \frac{1}{2} + it$ and $1 - \bar{s} = 1 - (\bar{s}) = 1 - (\overline{\frac{1}{2} + it}) = 1 - \frac{1}{2} + it = s$. Hence the wanted sequence becomes the constant zero sequence, which is convergent to zero as the constant zero sequence. □

3 Reformulating the problem

In the previous section we have shown the equivalence $\Re(s) = \frac{1}{2} \iff \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^{1-\bar{s}}} = 0$, since we have proven the two-sided implication by Lemma 1 and Lemma 2. By the transitivity of equivalence it is enough to show that the equivalence $\zeta(s) - \zeta(1-\bar{s}) = 0 \iff \Re(s) = \frac{1}{2}$ is false. For the needs of this section we shall introduce the following function:

$$\Phi(s) := 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s),$$

which we shall refer to. Notice that Riemann zeta-function has got no real zeroes in the critical strip, which makes the following argument possible.

Lemma 3. *$\Phi(s)$ and $\Phi(1-s)$ are multiplicative inverses.*

Proof. Let us express $\Phi(s)$ with respect to Riemann functional equation and use it again for $\zeta(1-s)$:

$$\zeta(s) = \Phi(s)\zeta(1-s) = \Phi(s)\Phi(1-s)\zeta(s) \iff \Phi(s)\Phi(1-s) = 1$$

□

Lemma 4. *Let $\epsilon > 0$ be the imaginary part of the closest to the real line point, where $\zeta(s) = 0$ in the critical strip. Then $\zeta(s) - \zeta(1-\bar{s}) = 0 \iff |\Phi(s)| = 1$ in the domain $\{s|\Re(s) \in (0,1), \Im(s) \in (-\epsilon, \epsilon)$.*

Proof. Firstly, let us prove the implication $\zeta(s) - \zeta(1-\bar{s}) = 0 \implies |\Phi(s)| = 1$. Since $\zeta(s) = \zeta(1-\bar{s})$ we may deduce that $|\zeta(s)| = |\zeta(1-\bar{s})| = |\zeta(1-s)|$ according to the Schwarz reflection principle. Now let us write the Riemann functional equation and take the absolute value of the both parts:

$$|\zeta(s)| = |\Phi(s)||\zeta(1-s)| = |\Phi(s)||\zeta(s)|.$$

Since $|\zeta(s)| \neq 0$ in the considered domain, divide both parts by $|\zeta(s)|$ and obtain $|\Phi(s)| = 1$.

Let us now prove the implication $|\Phi(s)| = 1 \implies \zeta(s) - \zeta(1-\bar{s}) = 0$. Let us write the Riemann functional equation for $\zeta(\bar{s})$ considering $\Phi(1-\bar{s}) = \Phi(s)$ and $\Phi(1-s) = \Phi(\bar{s})$ by the uniqueness of multiplicative inverses:

$$\zeta(s) = \Phi(s)\zeta(1-s), \tag{1}$$

$$\zeta(1-\bar{s}) = \Phi(s)\zeta(\bar{s}). \tag{2}$$

Subtract (2) from (1) and obtain and notice that $\Phi(s) = \frac{\zeta(s)}{\zeta(1-s)} = \frac{\zeta(1-\bar{s})}{\zeta(\bar{s})}$:

$$\zeta(s) - \zeta(1-\bar{s}) = \Phi(s)[\zeta(1-s) - \zeta(\bar{s})].$$

Notice that $|\Phi(s)| = 1 \implies \Phi(\bar{s}) = \frac{1}{\Phi(\bar{s})}$. Multiply both parts of this equation by $\sqrt{\Phi(\bar{s})}$ and denote $\Psi(s) := \sqrt{\Phi(\bar{s})}[\zeta(s) - \zeta(1-\bar{s})]$. Then this equality becomes the following:

$$\Psi(s) = -\Psi(\bar{s}). \tag{3}$$

From this we conclude that $\Psi(s)$ is purely imaginary with respect to the Schwarz reflection principle and the fact that complex conjugation preserves sums and products. Let us square both sides of this equality and obtain:

$$\Phi(\bar{s})[\zeta(s) - \zeta(1-\bar{s})]^2 = \Phi(s)[\zeta(1-s) - \zeta(\bar{s})]^2.$$

Multiply and divide the Right Hand Side by $\Phi^2(\bar{s})$

$$\Phi(\bar{s})[\zeta(s) - \zeta(1-\bar{s})]^2 = \Phi^3(s)[\zeta(s) - \zeta(1-\bar{s})]^2,$$

$$(\Phi^4(s) - 1)[\zeta(s) - \zeta(1-\bar{s})]^2 = 0.$$

Hence we are left to check the case when $\Phi^4(s) = 1$ since in other cases the statement is definitely true by this equation.

Notice that $H(s) := (\Phi^4(s) - 1)[\zeta(s) - \zeta(1-\bar{s})]^2$ is a continuous function as the product of linear combinations of continuous functions. Since the wanted equivalence is true for $\{s|\Phi^4(s) = 1\}$, which is the set of isolated points by the Theorem of Uniqueness, since in other case $\Phi(s)$ would be a constant, by the continuity of $H(s)$ we may deduce that $\zeta(s) - \zeta(1-\bar{s}) = 0$ in these points as well, since it is true for any arbitrary small neighbourhood of these points on the curve $|\Phi(s)| = 1$ by continuity. Therefore the equivalence is proven. □

4 Building the counter-example

Lemma 4 introduced us the equivalence $\zeta(s) - \zeta(1-\bar{s}) = 0 \iff |\Phi(s)| = 1$ for some neighbourhood of the real part of the critical strip, which does not contain zeroes of Riemann zeta-function. By the polar representation of the complex numbers we need to study the values of the variable, which can satisfy the equation $\Phi(s) = e^{i\alpha}$ for some $\alpha \in \mathbb{R}$. Let us study this equation from the geometric perspective.

Lemma 5. *The equation $\Phi(s) = e^{i\alpha}, \alpha \in \mathbb{R}$ defines at least one analytical curve on the complex plain $s(\alpha)$.*

Proof. Let us rewrite this equation in the following form:

$$F(\alpha, s) := e^{-i\alpha}\Phi(s) - 1 = 0.$$

Take the derivative of F with respect to s:

$$\tfrac{d}{ds}F(\alpha, s) = e^{-i\alpha}\Phi^{'}(s).$$

Since $\Phi(s)$ is a non-constant analytical function, the zeroes of $\Phi^{'}(s)$ would be a set of izolated points. Therefore in the neighbourhood of any point on the complex plain we can find a point, where $\Phi^{'}(s) \neq 0$. Hence by the Implicit Function Theorem we obtain the statement of the lemma. □

The next thing we would like to show is that the real component of this curve is non-constant.

Lemma 6. *Let $s(\alpha) = l(\alpha) + it(\alpha)$ be a curve, defined by the equation $\Phi(s) = e^{i\alpha}, \alpha \in (-\pi, \pi)$ such that $s(0) = \frac{1}{2}$. Then $l(\alpha) \neq const$.*

Proof. By the way of contradiction let us assume that $l(\alpha) = \frac{1}{2} = const$. Let us take the derivative of this equation with respect to α:

$$it^{'}(\alpha)\Phi^{'}(\tfrac{1}{2} + it(\alpha)) = ie^{i\alpha},$$

$$it^{'}(\alpha)\Phi^{'}(\tfrac{1}{2} + it(\alpha)) = i\Phi(\tfrac{1}{2} + it(\alpha)),$$

$$t^{'}(\alpha)\Phi^{'}(\tfrac{1}{2} + it(\alpha)) = \Phi(\tfrac{1}{2} + it(\alpha)).$$

Take $t(\alpha)$ to be an odd parameterization of the imaginary part of the curve, since $e^{-i\alpha}$ is conjugated to $e^{i\alpha}$ and it preserves conjugation by the Schwarz Reflection Principle. Then by the Schwarz reflection principle $\Im[\Phi(\frac{1}{2} + it(\alpha))]$ is an odd function. But the derivative of this function with respect to $it(\alpha)$, i.e. $\Im[\Phi^{'}(\frac{1}{2} + it(\alpha))]$ should be even as the derivative of an odd function. Therefore $t^{'}(\alpha)\Phi^{'}(\frac{1}{2} + it(\alpha))$ is an even function as the product of two even functions since $t^{'}(\alpha)$ is even as the derivative of an odd function. This means that $\Im[\Phi(\frac{1}{2} + it(\alpha))]$ is even and odd at the same time, which is only possible for the constant zero function. But $\Im[\Phi(\frac{1}{2} + it(\alpha)))] = \sin\alpha \neq 0$ constantly by the construction, hence we obtain a contradiction. This means that $l(\alpha) \neq const.$ ☐

Therefore we have proven that the counter-example exists, since the real component of such curve cannot be a constant function. This means that an analytical curve, parameterized in such a way, cannot lie on the critical line. Hence your equivalence is false as $\zeta(s) - \zeta(1-\bar{s}) = 0 \iff |\Phi(s)| = 1$, but it is not true that $|\Phi(s)| = 1 \iff \Re(s) = \frac{1}{2}$ by the Lemma 4 at least in the zero-free domain of the zeta-function. The Lemma 6 shows that in any arbitrary small subdomain, including a line segment of $\Re(s) = \frac{1}{2}$ such that a point $\frac{1}{2}$ belongs to this domain you can find a continuous set of counter-examples to your statement. But let us follow the rules of your contest and provide a numerical counter-example using these statements. For this reason we would need one more lemma.

Lemma 7. *Let $s(\alpha)$ be a curve from the Lemma 6. Then $s^{'}(\alpha) \neq 0$ and $\Phi^{'}(s(\alpha)) \neq 0$ for all $\alpha \in (-\pi, \pi)$.*

Proof. Take the derivative of both parts with respect to α of both parts of the equation $\Phi(s(\alpha)) = e^{i\alpha}$ and obtain:

$$s^{'}(\alpha)\Phi^{'}(s(\alpha)) = ie^{i\alpha}.$$

Since the exponential function is never zero and the product is zero if one of the multiples is zero, we obtain the statement of the lemma. ☐

The Lemma 7 guarantees the existence of at least one analytical inverse function to $\Phi(s)$ by the Lagrange Inversion Theorem. Let us pick an inverse and denote it by G with the property $G(1) = \frac{1}{2}$. Then it would preserve our curve and we could find the closed formula for the parameterization of the curve from Lemma 6 as follows:

$$s(\alpha) = G(e^{i\alpha}), \alpha \in (-\pi, \pi). \tag{4}$$

Notice that (4) defines an analytical curve without intersection with itself as a composition of injective functions on the wanted interval. Hence we are able to construct our counter-example as follows, which can be approximated with big enough computation powers:

$$s(\arg_{\alpha\in\Pi_s} \max |\Re s(\alpha) - \tfrac{1}{2}|),$$

where $\Pi_s := \{\alpha | s(\alpha) \in \{s | \Re(s) \in (0,1), \bar{\Im}(s) \in (-\epsilon, \epsilon)\}, \alpha \in (-\pi, \pi)\}$ and ϵ is the smallest positive imaginary part of the non-trivial zero of Riemann zeta-function.

5 A counter-example to the reasoning of your proof

Dear Aric, you've done an incredible work, but I have found one weak spot in your reasoning. I have provided you a full theoretical reasoning of what is wrong, but now I want to show you why such an extension of summation by the analytical continuation is not possible, since analytical continuation does not preserve the series structure. Let us study the example of $\zeta(0) = \frac{1}{2}$, but we are enough to know that it is defined. Following your notation we may obtain:

$$\zeta(0) = \sum_{n=1}^{\infty} 1 = 1 + \sum_{n=2}^{\infty} 1 = 1 + \sum_{n=1}^{\infty} 1 = 1 + \zeta(0),$$

$$\iff 0 = 1.$$

Hence we cannot extend the analytical continuation for series.