Uniqueness of Analytic Continuation: Necessary and Sufficient Conditions*

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Necessary and sufficient conditions for uniqueness of analytic continuation are investigated for a system of $m \geqslant 1$ first-order linear homogeneous partial differential equations in one unknown, with complex-valued $\mathscr{C}^\infty$ coefficients, in some connected open subset of $\mathbb{R}^k$, $k \geqslant 2$. The type of system considered is one for which there exists a real k-dimensional, $\mathscr{C}^\infty$, connected C–R submanifold M^k of $\mathbb{C}^n$, for $k, n \geqslant 2$, such that the system may be identified with the induced Cauchy–Riemann operators on M^k. The question of uniqueness of analytic continuation for a system of partial differential equations is thus transformed to the question of uniqueness of analytic continuation for C–R functions on the manifold $M^k \subset \mathbb{C}^n$. Under the assumption that the Levi algebra of M^k has constant dimension, it is shown that if the excess dimension of this algebra is maximal at every point, then M^k has the property of uniqueness of analytic continuation for its C–R functions. Conversely, under certain mild conditions, it is shown that if M^k has the property of uniqueness of analytic continuation for all $\mathscr{C}^\infty$ C–R functions, and if the Levi algebra has constant dimension on all of M^k, then the excess dimension must be maximal at every point of M^k.

1. INTRODUCTION

The problems considered in this paper arise out of both the fields of partial differential equations and several complex variables. In terms of partial differential equations we are interested in a system of m first-order linear homogeneous equations in one unknown, $m \geqslant 1$, with complex-valued $\mathscr{C}^\infty$ coefficients in $\mathbb{R}^k$, $k \geqslant 2$. We seek conditions under which a $\mathscr{C}^\infty$ solution on some connected set $A \subset \mathbb{R}^k$, which is zero on an arbitrary subset B with $\bar{B} \subset A$, has the property that it is identically zero on all of A. This is the study of uniqueness of analytic continuation for partial differential equations. Since our systems are linear, we have as a consequence that two $\mathscr{C}^\infty$ solutions u_1 and u_2 to a system with a $\mathscr{C}^\infty$ right-hand side which have the property that they agree on some open subset B must agree on all of A.

We will assume that our system (in $\mathbb{R}^k$) has the following property. There

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exists a real k-dimensional connected, $\mathscr{C}^\infty$ submanifold M^k (without boundary) of some $\mathbb{C}^n$, k, $n \geqslant 2$, such that the system can be identified with the induced Cauchy–Riemann operators on M^k from $\mathbb{C}^n$. Nirenberg [13, 14] has shown that there is a partial differential equation in three variables with $\mathscr{C}^\infty$ coefficients for which there does not exist an $M^3 \subset \mathbb{C}^2$ such that the partial differential equation is the Cauchy–Riemann operator on M^3. However, if our system has real-analytic coefficients or if it satisfies certain other weaker conditions (see [1] for some examples), then we can find the $M^k \subset \mathbb{C}^n$ we desire. It should be noted that Nirenberg's example in [14] does exhibit uniqueness of analytic continuation since all solutions to his partial differential equation are constants.

Therefore, we can change the question of uniqueness of analytic continuation for a homogeneous system of partial differential equations in $\mathbb{R}^k$ into the question of uniqueness of analytic continuation of C–R functions (functions satisfying the induced Cauchy–Riemann equations) on $M^k \subset \mathbb{C}^n$. Suppose then that f is a C–R function on M^k such that $f \equiv 0$ on some open subset U with $\overline{U} \subset M^k$. We wish to discover under what conditions we can be assured that f vanishes identically on all of M^k. We shall find that this depends on the excess dimension, e, of the Levi algebra. If the number e is as large as possible at every point of M^k, then we will show that we can extend f to a holomorphic function on some complex submanifold of $\mathbb{C}^n$, and then use the uniqueness of analytic continuation of holomorphic functions on this complex manifold to deduce that $f \equiv 0$ on M^k. We will also show that if M^k has the property of uniqueness of analytic continuation for all C–R functions which are $\mathscr{C}^\infty$, and if the Levi algebra has constant dimension on all of M^k, then the excess dimension must be maximal at every point of M^k. Our sufficiency result will be global on M^k, but our necessity result is essentially local in nature. The extensions of C–R functions on C–R manifolds has been considered in great generality in [11], and we shall need the results and methods of proof used there.

We remark that the results in [11] do not require M^k to be $\mathscr{C}^\infty$, but only sufficiently differentiable to enable the use of the Sobolev lemma. Similarly, our necessary and sufficient conditions will hold for $\mathscr{C}^m$ C–R functions, for some $m \geqslant 1$ which can be determined. However, for the sake of simplicity we will assume that M^k and its C–R functions are $\mathscr{C}^\infty$.

We will use some tools of several complex variables, particularly Lie brackets, Levi forms, and Levi algebras. These have been used before in the study of partial differential equations, most notably in the solvability work of Hörmander, Nirenberg, and Treves. Indeed, Hörmander's condition (H) can be stated precisely in these terms (see [9, 20]). It has been shown that a necessary condition for existence of distribution solutions for a large class of partial differential equations is that the first Lie bracket vanish. More recently, the use of successive Lie brackets has been exploited by Nirenberg and Treves

[15, 16], Treves [20], and Beals and Fefferman [2] to show sufficient conditions for solvability in a related class of partial differential equations, those satisfying condition (P) of Nirenberg and Treves.

Finally, Strauss and Treves [19] have shown uniqueness in the Cauchy problem for first-order linear partial differential equations satisfying condition (P). This result is related to ours in that the former implies uniqueness of analytic continuation in certain directions, whereas we get uniqueness of analytic continuation in all directions for the class of operators we consider.

Section 2 of this paper will contain definitions and examples. In Sections 3 and 4 we prove the principal sufficiency and necessity results, respectively. Section 5 contains other examples, results, and remarks.

2. Definitions and Examples

We begin by stating formally the definition of uniqueness of analytic continuation.

Definition 2.1. Suppose we have a system of $m(\geqslant 1)$ first-order linear homogeneous partial differential equations in one unknown with complex-valued $\mathscr{C}^\infty$ coefficients defined on an open connected subset A of $\mathbb{R}^k$, $k \geqslant 2$. This system has the property of *uniqueness of analytic continuation* on A if every $\mathscr{C}^\infty$ solution on A which vanishes identically on any arbitrary open subset of A must vanish identically on all of A.

In terms of several complex variables a more meaningful definition is the following.

Definition 2.1′. Let M^k be a real k-dimensional, connected, $\mathscr{C}^\infty$ manifold (possibly compact), which is a C–R submanifold of $\mathbb{C}^n$, with k and $n \geqslant 2$. Then M^k possesses the property of *uniqueness of analytic continuation* if every C–R function on M^k which vanishes identically on any arbitrary open subset of M^k must vanish on all of M^k.

The definitions of C–R manifolds and C–R functions will be given later. If the set A and the system in Definition 2.1 have the property that A can be embedded in some $\mathbb{C}^n$ as a real k-dimensional, connected, C–R manifold M^k, such that the system represents the induced Cauchy–Riemann operators on M^k, then Definition 2.1′ implies Definition 2.1.

Let M^k be a real k-dimensional $\mathscr{C}^\infty$ submanifold of $\mathbb{C}^n$, k, $n \geqslant 2$. For $p \in M^k$ we denote by $T_p(M^k)$ the real tangent space to M^k at p. The set $T(M^k)$ is the real tangent bundle to M^k with fiber $T_p(M^k)$, and the complex structure on $\mathbb{C}^n$ induces the almost complex tensor $J: T(\mathbb{C}^n) \to T(\mathbb{C}^n)$

defined by multiplication by $i = (-1)^{1/2}$. We define the *holomorphic tangent space* to M^k at p by

$$H_p(M^k) = T_p(M^k) \cap JT_p(M^k).$$

It is easily shown that $\max(k - n, 0) \leqslant \dim_{\mathbb{C}} H_p(M^k) \leqslant [k/2]$.

If $\dim_{\mathbb{C}} H_p(M^k)$ is a constant for every $p \in M^k$, then M^k is a *C–R manifold* in $\mathbb{C}^n$. These are the manifolds of interest to us because we can form the *holomorphic tangent bundle* $H(M^k)$ with fiber $H_p(M^k)$ for C–R manifolds. If M^k is a C–R manifold and $\dim_{\mathbb{C}} H_p(M^k) = \max(k - n, 0)$, then M^k is called a *generic manifold*. We have a *totally real manifold* M^k if $\dim_{\mathbb{C}} H_p(M^k) = 0$ for every $p \in M^k$. However, the manifolds we shall consider are those C–R manifolds which are not totally real; i.e., $0 < \dim_{\mathbb{C}} H_p(M^k) \leqslant [k/2]$, where [ ] denotes the greatest integer function.

Let x be a point in a C–R manifold $M^k \subset \mathbb{C}^n$, and let $X_1, \ldots, X_m$ be a local basis for $H(M^k)$ at x. A $\mathscr{C}^\infty$ function f on M^k is a *C–R function* at $x \in M^k$ if $\bar{X}_j f(y) = 0$, for all y near x, $j = 1, \ldots, m$, where $\bar{X}_j$, is the complex conjugate vector field to X_j. We then say that f satisfies the induced Cauchy–Riemann equations on M^k near $x \in M^k$. If f is a C–R function at every $x \in M^k$, then f is a C–R function on M^k. We can also define a C–R function in the following way [17]. Let $x \in M^k$ and $\rho_1, \ldots, \rho_j$ be real-valued $\mathscr{C}^\infty$ functions defined on a neighborhood N of x in $\mathbb{C}^n$ such that

(i) $d\rho_1 \wedge \cdots \wedge d\rho_j \neq 0$ on $M^k \cap N$,

(ii) $M^k \cap N = \{x \in N; \rho_1(x) = \cdots = \rho_j(x) = 0\}$.

Then there are $n - m = i$ of the ρ's, say $\rho_1, \ldots, \rho_i$, such that

$$\bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_i \neq 0 \qquad \text{near } x \text{ on } M^k.$$

Furthermore, a function $f \in \mathscr{C}^\infty(M^k)$ is a C–R function near x if and only if, for any $\tilde{f} \in \mathscr{C}^\infty(N)$ such that $\tilde{f} \mid M^k = f$,

$$\bar{\partial}f \wedge \bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_i = 0$$

on M^k near x.

Let f be a C–R function on a C–R manifold M^k such that $f \equiv 0$ on some arbitrary open subset U of M^k with $\bar{U} \subset M^k$. Our problem is to find necessary and sufficient conditions on M^k which insure that $f \equiv 0$. We shall work through a series of examples which indicate when unique analytic continuation can be expected, prior to proving our main result.

EXAMPLE 2.1. Let $M^3 \subset \mathbb{C}^2$ be given by the equations

$$z_1 = x_1,$$
$$z_2 = u_1 + iv_1 = w_1,$$

with $-1 < x_1 < 1$ and $| w_1 | < 1$. Here x_1, u_1, v_1 are coordinates for M^3, and z_1, z_2 are coordinates for $\mathbb{C}^2$. In this case we have that a basis section X of $H(M^3)$ is given by $\partial/\partial w_1$, by letting $\rho_1 = y_1$ in the definition of C–R functions. Thus, a C–R function f on M^3 must satisfy $\overline{X}f = \partial f/\partial \overline{w}_1 = 0$ at every point of M^k. Let U be the open subset of M^3 such that $-\frac{1}{2} < x_1 < \frac{1}{2}$ and $| w_1 | < \frac{1}{2}$. We define a C–R function f on M^3 by

$$
\begin{aligned}
f(z_1, z_2) &= 0, & -\tfrac{1}{2} < x_1 < \tfrac{1}{2}, \\
&= \exp(-1/(x_1 + \tfrac{1}{2})^2), & -1 < x_1 \leqslant -\tfrac{1}{2}, \\
&= \exp(-1/(x_1 - \tfrac{1}{2})^2), & \tfrac{1}{2} \leqslant x_1 < 1.
\end{aligned}
$$

We have that $f \equiv 0$ on U but not on all of M^3, and M^3 does not have the property of uniqueness of analytic continuation for C–R functions.

In order to understand Example 2.1 better let us examine the Levi form on a C–R manifold M^k. The *Levi form* is a map

$$
L(M^k): H(M^k) \to \frac{T(M^k) \otimes \mathbb{C}}{H(M^k) \otimes \mathbb{C}}
$$

defined by

$$
L_p(u) = \pi_p[Y, \overline{Y}]_p,
$$

where Y is a local cross section of $H(M^k)$ defined near p such that $Y_p = u$, $[\cdot, \cdot]$ denotes the Lie bracket, and π_p is the projection

$$
\pi_p: T_p(M^k) \otimes \mathbb{C} \to \frac{T_p(M^k) \otimes \mathbb{C}}{H_p(M^k) \otimes \mathbb{C}}.
$$

If the Levi form vanishes identically on M^k, we say that M^k is *Levi flat*.

In Example 2.1 we have that $[X, \overline{X}]_p = 0$ for every $p \in M^k$, and thus the Levi form vanishes identically on M^k. We shall see later that this vanishing is related to the fact that M^k does not have uniqueness of analytic continuation.

EXAMPLE 2.2. Let $M^4 \subset \mathbb{C}^3$ be given by the equations

$$
\begin{aligned}
z_1 &= x_1, \\
z_2 &= x_2 + iw_1\overline{w}_1, \\
z_3 &= u_1 + iv_1 = w_1,
\end{aligned}
$$

with $-1 < x_1, x_2 < 1$ and $| w_1 | < 1$. Here x_1, x_2, u_1, v_1 are coordinates on M^4 and z_1, z_2, z_3 are coordinates for $\mathbb{C}^3$. In this case a basis section X of $H(M^4)$ is given by

$$
-\frac{i}{2}\frac{\partial}{\partial w_1} + \overline{w}_1 \frac{\partial}{\partial z_2}.
$$

Thus a C–R function for M^4 must satisfy

$$\bar{X}f = \frac{i}{2}\frac{\partial f}{\partial \bar{w}_1} + w_1\frac{\partial f}{\partial \bar{z}_2} = 0$$

on M^4, as $\rho_1 = y_1$ and $\rho_2 = y_2 - w_1\bar{w}_1$. Let U be the open subset of M^4 satisfying $-\frac{1}{2} < x_1, x_2 < \frac{1}{2}$ and $|w_1| < \frac{1}{2}$. Define a C–R function f on M^4 by

$$
\begin{aligned}
f(z_1, z_2, z_3) &= 0, & -\tfrac{1}{2} < x_1 < \tfrac{1}{2} \\
&= \exp(-1/(x_1 + \tfrac{1}{2})^2), & -1 < x_1 \leqslant -\tfrac{1}{2} \\
&= \exp(-1/(x_1 - \tfrac{1}{2})^2), & \tfrac{1}{2} \leqslant x_1 < 1.
\end{aligned}
$$

We have that $f \equiv 0$ on U but not on all of M^4. Also, $[X, \bar{X}] = -(i/2)\{(\partial/\partial z_2) + (\partial/\partial \bar{z}_2)\}$, implying that the Levi form does not vanish anywhere on M^4. Thus we fail to have uniqueness of analytic continuation even though the Levi form vanishes nowhere.

Before continuing with Example 2.2, we need to give some more definitions. The *Levi algebra* of M^k, where M^k is a C–R manifold, is the Lie subalgebra of complex fields generated by sections of $H(M^k)$ and $\bar{H}(M^k)$, with $\bar{H}(M^k)$ the conjugate bundle of $H(M^k)$. The Levi algebra is denoted by $L(M^k)$, and we make the assumption that the dimension of $L(M^k)$ is constant. Then $L(M^k)$ is the algebra of sections of a vector bundle V, and $V \supset H(M^k) \oplus \bar{H}(M^k)$. Let

$$e = \text{fiber dim}_{\mathbf{C}} \frac{V}{H(M^k) \oplus \bar{H}(M^k)}.$$

This e is called the *excess dimension* of the Levi algebra. If M^k is a generic manifold, then $0 \leqslant e \leqslant 2n - k$, and if M^k is a C–R manifold with $l = \dim_{\mathbf{C}} H_p(M^k) - \max(k - n, 0)$, then $0 \leqslant e \leqslant 2n - k - 2l$. The first statement in the preceding sentence is obvious, and the second statement will follow from a proof in the next section. For a detailed discussion of the Levi algebra and the excess dimension, see [4]. Also notice that if $e \geqslant 1$, then the Levi form must not vanish on M^k.

If we examine Example 2.2 again we find that $[X, [X, \bar{X}]] = 0$ on all of M^4. This implies that the excess dimension e of M^4 is 1. Since M^4 is a generic manifold, the maximal possible excess dimension is 2. We will show that the fact that M^4 does not have the property of uniqueness of analytic continuation is related to the fact that its excess dimension is not maximal.

EXAMPLE 2.3.   Consider now the 3-manifold M^3 contained in $\mathbf{C}^2$ given by

$$
\begin{aligned}
z_1 &= x_1 + iw_1\bar{w}_1, \\
z_2 &= u_1 + iv_1 = w_1,
\end{aligned}
$$

with $-1 < x_1 < 1$ and $|w_1| < 1$. A basis section X of $H(M^3)$ is $-(i/2)(\partial/\partial w_1) + \overline{w}_1(\partial/\partial z_1)$. A C–R function f on M^3 must satisfy $\overline{X}f = (i/2)(\partial f/\partial \overline{w}_1) + w_1(\partial f/\partial \overline{z}_1) = 0$ on M^3, as $\rho_1 = y_1 - w_1\overline{w}_1$. Since $[X, \overline{X}] = -(i/2)\{(\partial/\partial z_1) + (\partial/\partial \overline{z}_1)\}$, all C–R functions on M^3 extend to holomorphic functions on a connected open subset W of $\mathbb{C}^2$ which contains M^3 in its boundary (see [8, 11, 12, 21]). If a C–R function f is zero in an open neighborhood U of the point $(0, 0, 0)$ in M^3, then the holomorphic function on W which agrees with f on M^3 can be shown to be zero on some open subset of W which is sufficiently close to the point $(0, 0, 0)$ in M^3. By the uniqueness of analytic continuation for holomorphic functions we have that this holomorphic function is identically zero in W, hence on M^3. Thus M^3 has the property of uniqueness of analytic continuation. We will see that this occurs because the excess dimension e is identically one on M^3, which is the maximum dimension possible.

Indeed, we will prove in the next section that if M^k is a real k-dimensional, connected, C–R, $\mathscr{C}^\infty$ submanifold of $\mathbb{C}^n$ such that the excess dimension of the Levi algebra on the manifold is maximal at every point, then M^k has the property of uniqueness of analytic continuation.

3. C–R Extension and Sufficiency Results

Let M^k be a C–R manifold and denote the C–R functions on M^k by $\mathrm{CR}(M^k)$. If M is another C–R manifold with M^k contained in the boundary of M, we say that M^k is *C–R extendible* to M if the map $\mathrm{CR}(M^k) \cap \mathrm{CR}(M) \to \mathrm{CR}(M^k)$ is onto. By this we mean that, given a C–R function f on M^k, there exists a C–R function f' on M such that $f' \mid M^k = f$.

Let D be the open unit disc in the complex plane given by $\{\zeta \in \mathbb{C}: |\zeta| < 1\}$. If T is the Cartesian product of q compact intervals on the real line, then we define a *q-real parameter family of analytic discs* as a continuous map F: $T \times \overline{D} \to \mathbb{C}^n$ such that $F(t, \zeta)$ is holomorphic for $\zeta \in D$ and for every fixed $t \in T$. If for some $t \in T, F(t, \zeta)$ is a constant, then we have a *degenerate analytic disc* corresponding to this value of t. In studying C–R extendibility these analytic discs prove to be very useful as the following known result indicates.

THEOREM 3.1 [11]. *Let M^k be a real k-dimensional $\mathscr{C}^\infty$ submanifold of $\mathbb{C}^n$. If M^k is C–R and if the Levi form does not vanish at some point $p \in M^k$, then there exists a $\mathscr{C}^m$ manifold $\tilde{M} \subset \mathbb{C}^n$ of real dimension $(k + 1)$ such that all C–R functions on M^k extend to C–R functions on $\tilde{M}$. The integer m can be chosen from the set $\{m : 1 \leqslant m < \infty\}$.*

We shall give an outline of the proof of this theorem in the generic case

only. Thus, we assume that M^k is a generic manifold and that M^k near p is given by the local equations

$$z_1 = x_1 + ih_1(x_1, ..., x_{2n-k}\, w_1, ..., w_{k-n}),$$
$$\vdots$$
$$z_{2n-k} = x_{2n-k} + ih_{2n-k}(x_1, ..., x_{2n-k}, w_1, ..., w_{k-n}), \qquad (3.1)$$
$$z_{2n-k+1} = u_1 + iv_1 = w_1,$$
$$\vdots$$
$$z_n = u_{k-n} + iv_{k-n} = w_{k-n},$$

where $x_1, ..., x_{2n-k}, u_1, v_1, ..., u_{k-n}, v_{k-n}$ are local coordinates for M^k vanishing at p, and $z_1, ..., z_n$ are coordinates on $\mathbb{C}^n$ also vanishing at p. The functions $h_1, ..., h_{2n-k}$ are real-valued and vanish to order 2 at p.

Using the Picard process of Bishop [3] and the Sobolev norm estimates of Wells [22], we can represent M^k near p in the form

$$z_1 = x_1^\infty + ih_1(x_1^\infty, ..., x_{2n-k}^\infty, w_1, ..., w_{k-n}),$$
$$\vdots$$
$$z_{2n-k} = x_{2n-k}^\infty + ih_{2n-k}(x_1^\infty, ..., x_{2n-k}^\infty, w_1, ..., w_{k-n}),$$
$$z_{2n-k+1} = w_1 = re^{i\theta}, \qquad (3.2)$$
$$z_{2n-k+2} = u_2 + iv_2 = w_2,$$
$$\vdots$$
$$z_n = u_{k-n} + iv_{k-n} = w_{k-n}.$$

The functions x_j^∞, $j = 1, ..., 2n - k$, are defined by $x_j^\infty = s_j - Th_j(x_1^\infty, ..., x_{2n-k}^\infty, re^{i\theta}, w_2, ..., w_{k-n})$, where Th_j is the Hilbert transform of $h_1(x_j^\infty, ..., x_{2n-k}^\infty, re^{i\theta}, w_2, ..., w_{k-n})$ taken with $s_1, ..., s_{2n-k}, r, w_2, ..., w_{k-n}$ fixed, where the s_j replace our original functions x_j. Given any positive integer m, the functions x_j^∞ can be taken to be $\mathscr{C}^m$ functions where defined. Then we have M^k near p given by a map $F: I^{2n-k} \times I_0 \times \partial D \times I^{2k-2n-2} \to \mathbb{C}^n$ defined by Eqs. (3.2) with $F = (F_1, ..., F_n) = (z_1, ..., z_n)$. We use the notation that $I = [-1, 1]$ and $I_0 = [0, 1]$. The variables, or more appropriately, parameters, are $s_1, ..., s_{2n-k}$ in I^{2n-k}, r in I_0, and $u_2, v_2, ..., u_{k-n}, v_{k-n}$ in $I^{2k-2n-2}$. Applying the Cauchy integral formula to the functions $F_1, ..., F_n$ we obtain a map $F': I^{2n-k} \times I_0 \times \bar{D} \times I^{2k-2n-2} \to \mathbb{C}^n$ such that $F' \mid I^{2n-k} \times I_0 \times \partial D \times I^{2k-2n-2} = F$, and F' is a holomorphic function in $\zeta \in D$ for $s_1, ..., s_{2n-k}, r, u_2, v_2, ..., u_{k-n}, v_{k-n}$ fixed. Thus we have a $(k - 1)$-real parameter family of analytic discs with boundaries in M^k near p, with the point p being a degenerate disc. Since the Levi form does not vanish at p we may arbitrarily assume that $\partial^2 h_j(p)/\partial w_1\, \partial \bar{w}_1 \neq 0$ for some j, $1 \leqslant j \leqslant 2n - k$. It is shown in [4] that if the parameter sets have been chosen sufficiently

small, then the interior of the family of analytic discs fills up a generic manifold $\tilde{M}$ of real dimension $(k + 1)$. This manifold $\tilde{M}$ is connected and simply connected.

Given a C–R function f on M^k there exists a C–R function f' on $\tilde{M}$ such that $f' \mid M^k = f$ near p. The function f' is obtained by applying the Cauchy integral formula to f on the boundary of each analytic disc. Proving that this function f' is actually C–R on $\tilde{M}$ is indeed very difficult and depends on approximations of f by holomorphic functions on certain "slices" of M^k. We need not do this here.

We are now ready to prove the main sufficiency result of this paper.

THEOREM 3.2. *Let M^k be a real k-dimensional, connected, C–R, $\mathscr{C}^\infty$ submanifold of $\mathbb{C}^n$, and let the excess dimension of the Levi algebra be maximal at every point. If f is a C–R function on M^k such that $f \equiv 0$ on U, U an open subset of M^k with $\overline{U} \subset M^k$, then $f \equiv 0$ on M^k.*

Proof. We have only to show that f is identically zero in an open neighborhood in M^k of an arbitrary boundary point q of U. So we just assume that M^k is this open neighborhood and U is an open subset of M^k with $\overline{U} \subset M^k$ and with q in the boundary of U.

The proof will be divided into two cases, the generic case and the C–R nongeneric case. So we first assume that M^k is a generic manifold, and that the excess dimension at every point is $2n - k$, which is the real codimension of M^k in $\mathbb{C}^n$.

The proof is by induction on the integers $2n - k$. If $2n - k = 0$, then M^k is just an open set in $\mathbb{C}^n$ and there is nothing to prove. So we assume that $2n - k = 1$. Given a positive integer m we can construct a $(k - 1)$-parameter family of analytic discs, which are defined by a $\mathscr{C}^m$ map, such that the boundaries of these discs fill up $B_\epsilon \cap M^k$, where B_ϵ is an open ball in $\mathbb{C}^n$ of radius ϵ about q. Here the point q takes the place of our point p in Eqs. (3.1), and hence is a degenerate disc. Since M^k is a $\mathscr{C}^\infty$ manifold there exists an open set, which we take to be B_ϵ, such that each point $q' \in B_\epsilon$ has the property that there exists a $(k - 1)$-parameter family of analytic discs, constructed with q' as the degenerate disc, having their boundaries filling up $B_{\epsilon/2}(q') \cap M^k$, where $B_{\epsilon/2}(q')$ is an open ball in $\mathbb{C}^n$ with radius $\epsilon/2$ and center q'. We choose a point $p \in M^k \cap U$ such that the distance from p to q in $\mathbb{C}^n$ is less than $\epsilon/4$. So we can construct a $(k - 1)$ parameter family of analytic discs at p, as in (3.1), such that these discs have $\mathscr{C}^m$ structure and such that the point q is on the boundary of one of the discs. Any C–R function g on M^k extends to a C–R function g' on the interior of these discs by Theorem 3.1.

The assumption that $2n - k = 1$ implies that $k = 2n - 1$. Thus the interior of the family of analytic discs, which we denote by $\tilde{M}$, is simply an

open connected set in $\mathbb{C}^n$ with M^k near p in its boundary. Since $p \in U$ and $f \equiv 0$ in U, we find that the extended C–R function f' is zero on those discs whose boundaries are sufficiently close to p. By the uniqueness of analytic continuation for holomorphic functions in $\mathbb{C}^n$ we have that $f' \equiv 0$ on $\tilde{M}$, hence $f \equiv 0$ on the boundaries of the analytic discs. Here we have used the fact that a C–R function on an open set in $\mathbb{C}^n$ is a holomorphic function in the set. Thus we have shown that $f \equiv 0$ on a neighborhood of q in M^k.

Now we assume the theorem is true if $2n - k \leqslant t - 1$, and show that it is true for $2n - k = t$, with t an integer greater than 1. We take p and q as above and suppose that we have constructed the analytic discs at p as before. All C–R functions on M^k extend to C–R functions on $\tilde{M}$ by Theorem 3.1. The function f' which extends f has the property that it is identically zero on an open subset of $\tilde{M}$. Since $\tilde{M}$ is a generic manifold and the excess dimension of $\tilde{M}$ is $t - 1$ (see [4]), then by our inductive hypothesis $f' \equiv 0$ on $\tilde{M}$. This implies that $f \equiv 0$ in an open neighborhood of q in M^k, and we have completed the proof in the generic case.

Now we assume that M^k is a nongeneric, C–R manifold in $\mathbb{C}^n$. Let $p \in M^k$ and take $l = \dim_{\mathbb{C}} H_p(M^k) - \max(k - n, 0) > 0$. Then the local equations for M^k near p are (in the case that $k \geqslant n$, with a similar argument if $k < n$)

$$
\begin{aligned}
z_1 &= x_1 + ih_1(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}), \\
&\vdots \\
z_{2(n-l)-k} &= x_{2(n-l)-k} + ih_{2(n-l)-k}(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}) \\
z_{2(n-l)-k+1} &= u_1 + iv_1 = w_1, \\
&\vdots \\
z_{n-l} &= u_{k-n+l} + iv_{k-n+l} = w_{k-n+l}, \\
z_{n-l+1} &= g_1(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}), \\
&\vdots \\
z_n &= g_l(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}),
\end{aligned}
\tag{3.3}
$$

where $x_1, \ldots, x_{2(n-l)-k}, u_1, v_1, \ldots, u_{k-n+l}, v_{k-n+l}$ are local coordinates for M^k vanishing at p, and $z_1, \ldots, z_n$ are coordinates on $\mathbb{C}^n$ also vanishing at p. The functions $h_1, \ldots, h_{2(n-l)-k}$ and the functions $g_1, \ldots, g_l$ vanish to order 2 at p. It is shown in [10] that M^k is a C–R manifold if and only if $g_1, \ldots, g_l$ are C–R functions on the *associated generic manifold* M_0^k in $\mathbb{C}^{n-l}$ given by the equations

$$
\begin{aligned}
z_1 &= x_1 + ih_1(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}), \\
&\vdots \\
z_{2(n-l)-k} &= x_{2(n-l)-k} + ih_{2(n-l)-k}(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}), \\
z_{2(n-l)-k+1} &= u_1 + iv_1 = w_1, \\
&\vdots \\
z_{n-l} &= u_{k-n+l} + iv_{k-n+l} = w_{k-n+l}.
\end{aligned}
\tag{3.4}
$$

It is true that the excess dimension for M^k is the same as that of $M_0{}^k$, implying that the maximum possible value for e is $2n - k - 2l$.

Let us examine another generic manifold M' in $\mathbb{C}^n$ with local equations given by

$$z_1 = x_1 + ih_1(x_1, ..., x_{2(n-l)-k}, w_1, ..., w_{k-n+l}),$$
$$\vdots$$
$$z_{2(n-l)-k} = x_{2(n-l)-k} + ih_{2(n-l)-k}(x_1, ..., x_{2(n-l)-k}, w_1, ..., w_{k-n+l}),$$
$$z_{2(n-l)-k+1} = u_1 + v_1 = w_1,$$
$$\vdots \tag{3.5}$$
$$z_{n-l} = u_{k-n} + iv_{k-n} = w_{k-n},$$
$$z_{n-l+1} = x_{n-l+1} + ig_{1,2}(x_1, ..., x_{2(n-l)-k}, w_1, ..., w_{k-n+l}),$$
$$\vdots$$
$$z_n = x_n + ig_{l,2}(x_1, ..., x_{2(n-l)-k}, w_1, ..., w_{k-n+l}),$$

where $g_j = g_{j,1} + ig_{j,2}$ in Eqs. (3.3) for $j = 1, ..., l$. Now we want to construct the boundaries for the analytic discs with respect to w_1 of the manifolds described in Eqs. (3.3), (3.4), and (3.5). These are

$$z_1 = x_1{}^\infty + ih_1(x_1{}^\infty, ..., x_{2(n-l)-k}^\infty, w_1, ..., w_{k-n+l}),$$
$$\vdots$$
$$z_{2(n-l)-k} = x_{2(n-l)-k}^\infty + ih_{2(n-l)-k}(x_1{}^\infty, ..., x_{2(n-l)-k}^\infty, w_1, ..., w_{k-n+l}),$$
$$z_{2(n-l)-k+1} = w_1 = re^{i\theta},$$
$$z_{2(n-l)-k+2} = u_2 + iv_2 = w_2,$$
$$\vdots \tag{3.6}$$
$$z_{n-l} = u_{k-n+l} + iv_{k-n+l} = w_{k-n+l},$$
$$z_{n-l+1} = g_1(x_1{}^\infty, ..., x_{2(n-l)-k}^\infty, w_1, ..., w_{k-n+l}),$$
$$\vdots$$
$$z_n = g_l(x_1{}^\infty ..., x_{2(n-l)-k}^\infty, w_1, ..., w_{k-n+l}),$$

for M^k,

$$z_1 = x_1{}^\infty + ih_1(x_1{}^\infty, ..., x_{2(n-l)-k}^\infty, w_1, ..., w_{k-n+l}),$$
$$\vdots$$
$$z_{2(n-l)-k} = x_{2(n-l)-k}^\infty + ih_{2(n-l)-k}(x_1{}^\infty, ..., x_{2(n-l)-k}^\infty, w_1, ..., w_{k-n+l}),$$
$$z_{2(n-l)-k+1} = w_1 = re^{i\theta},$$
$$z_{2(n-l)-k+2} = u_2 + iv_2 = w_2, \tag{3.7}$$
$$\vdots$$
$$z_{n-l} = u_{k-n+l} + iv_{k-n+l} = w_{k-n+l},$$

for $M_0{}^k$, and

$$
\begin{aligned}
z_1 &= x_1{}^\infty + ih_1(x_1{}^\infty,..., x_{2(n-l)-k}^\infty, w_1,..., w_{k-n+l}), \\
&\vdots \\
z_{2(n-l)-k} &= x_{2(n-l)-k}^\infty + ih_{2(n-l)-k}(x_1{}^\infty,..., x_{2(n-l)-k}^\infty, w_1,..., w_{k-n+l}), \\
z_{2(n-l)-k+1} &= w_1 = re^{i\theta}, \\
z_{2(n-l)-k+2} &= u_2 + iv_2 = w_2, \\
&\vdots \\
z_{n-l} &= u_{k-n} + iv_{k-n} = w_{k-n}, \\
z_{n-l+1} &= x_{n-l+1} - Tg_{1,2}(x_1{}^\infty,..., x_{2(n-l)-k}^\infty, w_1,..., w_{k-n+l}) \\
&\quad + ig_{1,2}(x_1{}^\infty,..., x_{2(n-l)-k}^\infty, w_1,..., w_{k-n+l}), \\
&\vdots \\
z_n &= x_n - Tg_{l,2}(x_1{}^\infty,..., x_{2(n-l)-k}^\infty, w_1,..., w_{k-n+l}) \\
&\quad + ig_{l,2}(x_1{}^\infty,..., x_{2(n-l)-k}^\infty, w_1,..., w_{k-n+l}),
\end{aligned}
\tag{3.8}
$$

for M'. The discs with boundaries given in (3.6) form a $(k + 1)$-dimensional manifold $\tilde{M}$ with M^k in its boundary, the discs with boundaries given in (3.7) form a $(k + 1)$-dimensional generic manifold $\tilde{M}_0$ with $M_0{}^k$ in its boundary, and the discs with boundaries given in (3.8) form a $(k + l + 1)$-dimensional generic manifold $\tilde{M}'$.

What we need to know is that $\tilde{M}$ is a C–R manifold in $\mathbb{C}^n$, and that all C–R functions on M^k extend to C–R functions on $\tilde{M}$. The functions $g_1,..., g_l$ are C–R on the generic manifold $M_0{}^k$ and thus extend to C–R functions on $\tilde{M}_0$ by Theorem 3.1. Thus we have that $\tilde{M}$ is a C–R manifold, since $\tilde{M}_0$ is the associated generic manifold for $\tilde{M}$. In [11] it is shown that all C–R functions on M^k extend to C–R functions on the manifold M', and hence to C–R functions on $\tilde{M}'$, a generic manifold. But $\tilde{M}$ is a submanifold of $\tilde{M}'$, and hence all C–R functions on M^k extend to C–R functions on the C–R manifold $\tilde{M}$.

The proof in the C–R, nongeneric case is by induction as in the generic case. The only change is that if the maximal excess dimension is one, then $\tilde{M}$ is a complex submanifold of $\mathbb{C}^n$ of complex dimension $n - l$ instead of an open set in $\mathbb{C}^n$ as in the generic case. Thus we use the uniqueness of analytic continuation for complex manifolds instead of for open subsets of $\mathbb{C}^n$. Q.E.D.

4. Necessity Results

The theorem we will prove later in this section is given as follows.

THEOREM 4.1. *Let M^k be a real k-dimensional, connected, C–R, $\mathscr{C}^\infty$ submanifold of $\mathbb{C}^n$, and let the excess dimension of the Levi algebra be constant at*

every point. If M^k has the property of uniqueness of analytic continuation for its C–R functions, then the excess dimension e must be maximal at every point of M^k.

This is the converse of Theorem 3.2. However, we cannot prove the necessity result above without making the following additional assumptions on the type of uniqueness of we are considering.

A. If M^k has the property of uniqueness of analytic continuation, then so does any arbitrary connected open subset of M^k. In other words global uniqueness implies local uniqueness.

B. If M^k and $\tilde{M}$ are taken as in Section 3, and if $\tilde{M} \cup M^k$ has uniqueness of analytic continuation, then so does any connected open subset $\mathcal{O}$ of $\tilde{M}$. By $\tilde{M} \cup M^k$ having uniqueness of analytic continuation we mean that any function in $\mathscr{C}^m(\tilde{M} \cup M^k)$ which is C–R on both $\tilde{M}$ and M^k and which is zero on an arbitrary open set in $\tilde{M}$, is identically zero on $\tilde{M} \cup M^k$.

Since our proof will be local in nature, we remark that assumption A is not needed if one is interested only in a local result.

Let M^k be a real k-dimensional, connected, C–R, $\mathscr{C}^\infty$ manifold embedded in $\mathbb{C}^n$. Let p be a point in M^k such that the Levi form on M^k does not vanish at p. As in Section 3, there exists a $(k-1)$-parameter family of analytic discs with boundaries in M^k near p, such that the interiors of these discs fill up a connected C–R manifold $\tilde{M}$ of real dimension $(k+1)$. In the following lemma we assume that M^k consists of the connected open neighborhood of p in M^k filled up by the boundaries of the discs giving us $\tilde{M}$.

LEMMA 4.2. *If M^k has the property of uniqueness of analytic continuation, then $\tilde{M} \cup M^k$ also has this property.*

Proof. Let f be a C–R function on both $\tilde{M}$ and M^k, and let U be an open subset of $\tilde{M}$ with $f \equiv 0$ on U. Since f is holomorphic on each of the analytic discs we have that $f \equiv 0$ on the closure of those discs which intersect U. Hence there exists an open subset of M^k, which is contained in the boundaries of such discs, with $f \equiv 0$ on this open set. Since M^k has the property of uniqueness of analytic continuation, it follows that f is identically zero on M^k. By the maximum principle on the analytic discs, $f \equiv 0$ on $\tilde{M} \cup M^k$, and the lemma is proved. Q.E.D.

In the following lemma we assume that $2(n-l) - k \geqslant 1$, because if $2(n-l) - k = 0$, then M^k is a complex manifold, which has the property of uniqueness of analytic continuation. Here $l = \dim_{\mathbb{C}} H_p(M^k) - \max(k-n, 0)$, as before.

LEMMA 4.3. *Let M^k be a real k-dimensional, connected, C–R, $\mathscr{C}^\infty$ submanifold of $\mathbb{C}^n$ such that the Levi form vanishes on an open subset of M^k. Then M^k does not have uniqueness of analytic continuation.*

Proof. Let $\mathcal{O}$ be a connected open subset of M^k described by Eqs. (3.3), and suppose the Levi form vanishes identically on $\mathcal{O}$. Then we need only show that $\mathcal{O}$ does not have uniqueness of analytic continuation, and then apply assumption (A). It follows from the work of Sommer and Wells (see [18, 22]) that $\mathcal{O}$ can by given by the equations

$$
\begin{aligned}
z_1 &= \alpha_1(t_1,...,t_{2(n-l)-k}, \gamma_1,...,\gamma_{k-n+l}), \\
&\vdots \\
z_n &= \alpha_n(t_1,...,t_{2(n-l)-k}, \gamma_1,...,\gamma_{k-n+l}),
\end{aligned}
\tag{4.1}
$$

where $t_1,...,t_{2(n-l)-k}$ are real parameters, $\gamma_1,...,\gamma_{k-n+l}$ are complex variables, and $\alpha_1,...,\alpha_n$ are holomorphic in $\gamma_1,...,\gamma_{k-n+l}$ for $t_1,...,t_{2(n-l)-k}$ fixed. We assume that $-1 < t_1,...,t_{2(n-l)-k} < 1$ and $|\gamma_1|,...,|\gamma_{k-n+l}| < 1$.

The function defined by

$$
\begin{aligned}
f(z_1,...,z_n) &= 0, & -\tfrac{1}{2} < t_1 < \tfrac{1}{2}, \\
&= \exp(-1/(t_1 + \tfrac{1}{2}))^2, & -1 < t_1 \leqslant -\tfrac{1}{2}, \\
&= \exp(-1/(t_1 - \tfrac{1}{2}))^2, & \tfrac{1}{2} \leqslant t_1 < 1,
\end{aligned}
$$

is certainly C–R on $\mathcal{O}$ since it is holomorphic in the variables $\gamma_1,...,\gamma_{k-n+l}$. It is identically zero on an open subset of $\mathcal{O}$ without being zero on all of $\mathcal{O}$, and $\mathcal{O}$ does not have analytic continuation.   Q.E.D.

Now we are ready to prove the main necessity result of this paper, Theorem 4.1.

Proof. We consider only the cases in which $2(n-l) - k \geqslant 1$, and the proof will be by induction on $2(n-l) - k = k - 2\dim_{\mathbb{C}} H_p(M^k)$. If $2(n-l) - k = 1$, the maximum number possible for the excess dimension is 1. If $e \equiv 0$ on M^k, then M^k cannot have uniqueness of analytic continuation by Lemma 4.3, a contradiction.

Let t be an integer greater than 1, and assume the statement of the theorem is true for $2(n-l) - k = t - 1$. We will show that if M^k is a manifold with the maximum possible excess dimension equal to t, and with M^k having uniqueness of analytic continuation, then e, the excess dimension of $\mathscr{L}(M^k)$, is t. By using analytic discs we can find the manifold $\tilde{M}$ in Lemma 4.2 if there exists a point $p \in M^k$ where $e > 0$. However, if $e \equiv 0$ for every point in M^k, then we have a contradiction by using the fact that the Levi form vanishes on M^k if e vanishes on M^k. Applying Lemma 4.2, $\tilde{M} \cup M^k$ must

have the property of uniqueness of analytic continuation. By assumption B, the manifold $\tilde{M}$ itself must also have this property. Using arguments as in [5] it can be shown that the excess dimension of $\mathscr{L}(\tilde{M})$ is 1 less than the excess dimension of $\mathscr{L}(M^k)$, and this is also true for the maximum possible excess dimensions of $\mathscr{L}(M^k)$ and $\mathscr{L}(\tilde{M})$. Thus the maximum possible excess dimension of $\mathscr{L}(\tilde{M})$ is $t - 1$. By our inductive hypothesis, the excess dimension of $\mathscr{L}(\tilde{M})$ must be $t - 1$, since $\tilde{M}$ has uniqueness of analytic continuation. Hence the excess dimension e of $\mathscr{L}(M^k)$ is exactly t.   Q.E.D.

The following corollary implies that assumption B is not required if only real hypersurfaces in $\mathbb{C}^n$ are considered.

COROLLARY 4.4.   *Assumption B is not needed if $k - 2 \dim_{\mathbb{C}} H_p(M^k) = 1$ for $p \in M^k$.*

Proof.   In this case the maximum number possible for e is 1, and if $e \equiv 0$, we just use Lemma 4.3 and not Lemma 4.2.   Q.E.D.

5. OTHER EXAMPLES, RESULTS, AND REMARKS

Thus far in this paper we have considered only those C–R manifolds M^k which have constant dimension for their Levi algebras at every point. It is important to ask if there exist manifolds $M^k \subset \mathbb{C}^n$ which do not have the property that their Levi algebras have constant dimension, but which still have uniqueness of analytic continuation for their C–R functions. If M^k is a hypersurface in $\mathbb{C}^n$ (i.e., $k = 2n - 1$), then the dimension of the Levi algebra is zero if and only if the Levi form vanishes, and the largest possible dimension is 1. In the case of the hypersurface, we will consider examples for which the Levi form vanishes at some points, but not at others.

EXAMPLE 5.1.   Consider the 3-manifold M^3 in $\mathbb{C}^2$ given by the equations

$$z_1 = x_1 + i(w_1 \bar{w}_1)^2,$$
$$z_2 = u_1 + iv_1 = w_1,$$

with $-1 < x_1 < 1$ and $|w_1| < 1$. We find $X = -(i/2)(\partial/\partial w_1) + 2(w_1 \bar{w}_1^2)(\partial/\partial z_1)$ is a basis section of $H(M^3)$ and

$$[X, \bar{X}] = -2i w_1 \bar{w}_1 \cdot \{(\partial/\partial z_1) + (\partial/\partial \bar{z}_1)\}.$$

Thus the Levi form does not vanish on M^3 except on that set of points where $w_1 = 0$. This set of points S forms a real 1-dimensional submanifold of M^3. If U is any open subset of M^3, then $U - S$ contains an open set in the con-

nected manifold $M^3 - S$. If f is a C–R function on M^3 with $f \equiv 0$ on U, then applying Theorem 3.2 we find that $f \equiv 0$ on $M^3 - S$. By continuity $f \equiv 0$ on M^k, and we have the desired result.

We doubt that this example can be generalized in the following sense. Let M^{2n-1} be a real hypersurface in $\mathbb{C}^n$ such that the Levi form vanishes at most on a nowhere dense subset of M^{2n-1}; then M^{2n-1} has the property of uniqueness of analytic continuation. For the M^3 in the above example, we notice that the Levi form vanishes only on a subset of dimension 1. If we have a manifold M^{2n-1} and a subset S of M^{2n-1} on which the Levi form vanishes, and this set S disconnects M^{2n-1}, then finding necessary and sufficient conditions for M^{2n-1} to have uniqueness of analytic continuation seems to be very difficult.

An interesting problem is illustrated by the following example.

EXAMPLE 5.2. Let M^3 be the $\mathscr{C}^\infty$ submanifold of $\mathbb{C}^2$ defined by the equations

$$z_1 = x_1 + ih_1(x_1, w_1),$$
$$z_2 = u_1 + iv_1 = w_1,$$

with $-1 < x_1 < 1$ and $|w_1| < 1$, and where h_1 is the function given by

$$h(x_1, w_1) = 0, \qquad\qquad |w_1| \leqslant \hat{r}$$

$$= \exp\left(\frac{-1}{\hat{r}^2 - w_1 \bar{w}_1}\right), \qquad \hat{r} < |w_1| < 1.$$

Here $\hat{r}$ is fixed so that $0 < \hat{r} < 1$. We can put in the analytic discs by applying the Cauchy integral formula to the equations

$$z_1 = x_1 + ih_1(x_1, re^{i\theta}),$$
$$z_2 = u_1 + iv_1 = w_1 = re^{i\theta},$$

since no Picard process and Sobolev norm estimates are needed in this case (h_1 is a constant function of x_1). These analytic discs fill up an open set W in $\mathbb{C}^2$ described by $\{(z_1, z_2) \in \mathbb{C}^2; |z_2| < 1, y_1 - h(x_1, w_1) > 0\}$. It can be shown that all holomorphic functions in a neighborhood of M^3 in $\mathbb{C}^2$ extend to holomorphic functions on W, that all C–R functions on M^3 can be approximated uniformly on M^3 by holomorphic functions in a neighborhood of M^3, and hence all C–R functions on M^3 extend to holomorphic functions on W (see [17, 21]). If a C–R function on M^3 vanishes on any open subset of M^3, then it must vanish identically on all of M^3 by arguments given in [7, 14]. However, if U is an open subset of M^3 on which the Levi form vanishes (for instance a sufficiently small open neighborhood of the origin), then U

does not have uniqueness of analytic continuation. Hence this manifold M^3 has the property of uniqueness of analytic continuation for its C–R functions, but not every open subset of M^3 has this property. This example, which has a nonconstant dimension for its Levi algebra, verifies the following statement: Given a real hypersurface $M^{2n-1} \subset \mathbb{C}^n$ which has uniqueness of analytic continuation, it is not always true that every open subset of M^{2n-1} has uniqueness of analytic continuation.

For compact connected real hypersurfaces in $\mathbb{C}^n$ we find that a remarkable phenomenon occurs. The following result is the basis for such a phenomenon (see [8]).

THEOREM 5.3. *Let Ω be a bounded open set in $\mathbb{C}^n$, $n \geqslant 2$, such that the complement of the closure of Ω is connected and $\partial\Omega$ is a $\mathscr{C}^\infty$ submanifold of $\mathbb{C}^n$. If u is a C–R function on $\partial\Omega$, then there exists a holomorphic function u' in Ω such that $u' \mid \partial\Omega = u$.*

Using Theorem 5.3 we can prove the following global theorem.

THEOREM 5.4. *Let Ω and $\partial\Omega$ be as in Theorem 4.1. If f is a C–R function on $\partial\Omega$ which is identically zero on some open set in $\partial\Omega$ then $f \equiv 0$.*

Proof. The holomorphic extension f' vanishes on an open subset of Ω and so is identically zero (see [7, 14]). Q.E.D.

REMARK 5.5. Harvey and Lawson [6] have proved a theorem analogous to Theorem 5.3, but with $\partial\Omega$ replaced by a compact connected odd-dimensional $\mathscr{C}^\infty$ submanifold $M^k \subset \mathbb{C}^n$ with $\dim_\mathbb{C} H_p(M^k)$ maximal for every $p \in M^k$, and with Ω replaced by a $(k+1)$-dimensional complex manifold with possible singularities. For such sets there is probably an analog to Theorem 5.4.

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